# ЛААГ - I КОЛ - 2024

## ГРУПЕ ЗАДАТАКА ЛАПЛАС И КРАМЕР

$\boxed{1}$

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & \text{групе} & \\ \end{array}\;\begin{array}{cc} \text{лаплас} & \text{крамер} \end{array}$$

$$\left[\begin{array}{cccc|c|c} \boxed{1} & -1 & 2 & 3 & 8 & 18 \\ -1 & 2 & 1 & -5 & -7 & -20 \\ 5 & -8 & -1 & 17 & 31 & 76 \\ 1 & 1 & 11 & 7 & 21 & 52 \end{array}\right] \sim \left[\begin{array}{cccc|c|c} 1 & -1 & 2 & 3 & 8 & 18 \\ 0 & \boxed{1} & 3 & -2 & 1 & -2 \\ 0 & -3 & -11 & 2 & -9 & -14 \\ 0 & 2 & 9 & 4 & 13 & 34 \end{array}\right] (3)\,(-2)$$

$$\sim \left[\begin{array}{cccc|c|c} 1 & -1 & 2 & 3 & 8 & 18 \\ 0 & 1 & 3 & -2 & 1 & -2 \\ 0 & 0 & \boxed{-2} & -4 & -6 & -20 \\ 0 & 0 & 3 & 8 & 11 & 38 \end{array}\right] \left(-\tfrac{1}{2}\right) \tfrac{3}{2} \sim \left[\begin{array}{cccc|c|c} 1 & -1 & 2 & 3 & 8 & 18 \\ 0 & 1 & 3 & -2 & 1 & -2 \\ 0 & 0 & 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & \boxed{2} & 2 & 8 \end{array}\right] \tfrac{1}{2}\;\; -1 \;\; -3 \;\; -\tfrac{3}{2}$$

$$\sim \left[\begin{array}{cccc|c|c} 1 & -1 & 2 & 0 & 5 & 6 \\ 0 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & \boxed{1} & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{array}\right] -3 \;\; -2 \sim \left[\begin{array}{cccc|c|c} 1 & -1 & 0 & 0 & 3 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{array}\right] \Rightarrow$$

ГРУПА ЛАПЛАС

$(x_1, x_2, x_3, x_4) = (3, 0, 1, 1)$ ← ДАТУМ КОЛОКВИЈУМА

ГРУПА КРАМЕР

$(x_1, x_2, x_3, x_4) = (2, 0, 2, 4)$ ← ГОДИНА КОЛОКВИЈУМА ■

$\boxed{2}$ ЛАПЛАС

ⓐ КОНТРАПРИМЕР $u_1 = (1, 2, 2, -1) \in U_1$, $u_2 = (1, 3, 3, -1) \in U_1$

али $u_1 + u_2 = (2, 5, 5, -2) \notin U_1$ јер $5 \neq 2 \cdot 5$ $\Rightarrow \boxed{U_1 \not\leq \mathbb{R}^4}$

ⓑ $U_2 = \left\{(a,b,c,d) \,\middle|\, \begin{matrix} a+b+d=0 \\ -a+b+d=0 \end{matrix}\right\} = \left\{(a,b,c,d) \,\middle|\, \begin{matrix} a+b+d=0 \\ 2b=0 \end{matrix} \Rightarrow d=-a \right\}$

$= \{(a, 0, c, -a) \mid a, c \in \mathbb{R}\} = \boxed{\mathcal{L}((1,0,0,-1), (0,0,1,0))} \Rightarrow \boxed{\dim U_2 = 2}$

$U_3$: $\begin{bmatrix} 1 & 0 & -1 & 2 \\ -2 & 0 & 1 & 1 \\ 0 & 0 & -2 & 10 \\ 1 & 0 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -2 & 10 \\ 0 & 0 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} U_3 = \mathcal{L}((1,0,-1,2), (-2,0,1,1)) \\ \dim U_3 = 2 \end{matrix}}$

ⓥ $\underbrace{(a,b,c,d)}_{\in \mathbb{R}^4} = \underbrace{x(1,0,0,-1) + y(0,0,1,0)}_{\in U_2} + \underbrace{z(1,0,-1,2) + t(-2,0,1,1)}_{\in U_3}$

$(a, \underset{\sim}{b}, c, d) = (x, \underset{\sim}{0}, y, -x) + (z - 2t, \underset{\sim}{0}, -z + t, 2z + t)$

Ако посматрамо 2. коорд $\Rightarrow b = 0$ што не мора важити.

Дакле, нпр $(0,1,0,0) \in \mathbb{R}^4$ се не може записати у облику $u_2 + u_3$, где $u_2 \in U_2$ и $u_3 \in U_3$ $\Rightarrow \boxed{\mathbb{R}^4 \neq U_2 + U_3}$ па није ни директна сума ■

КРАМЕР

ⓐ КОНТРАПРИМЕР $w_1 = (1, -1, 2, 2) \in W_1$, $w_2 = (1, -1, 3, 3) \in W_1$

али $w_1 + w_2 = (2, -2, 5, 5) \notin W_1$ јер $5 \neq 2 \cdot 5$

ⓑ $W_2 = \mathcal{L}((1,0,0,-1), (0,1,0,0))$, $\dim W_2 = 2$ (само замењени c и b)

$W_3 = \mathcal{L}((1,-1,0,2), (-2,1,0,1))$, $\dim W_3 = 2$

ⓥ $\mathbb{R}^4 \ni (0,0,1,0) \neq w_2 + w_3$, $w_2 \in W_2$, $w_3 \in W_3$ $\Rightarrow \mathbb{R}^4 \neq W_2 + W_3$ па није ни директна сума ■

$\boxed{3}$ ЛАПЛАС $\quad L: M_2(\mathbb{R}) \to M_2(\mathbb{R})$, $L(X) = AX + X^T A$, $A = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

ⓐ 1) $L(X+Y) = A(X+Y) + (X+Y)^T A = AX + AY + (X^T + Y^T)A$
$= AX + X^T A + AY + Y^T A = L(X) + L(Y)$

2) $L(\alpha X) = A(\alpha X) + (\alpha X)^T A = \alpha(AX + X^T A) = \alpha L(X)$ $\Big\}$ $\underline{L \text{ линеарн}}$

ⓑ $\boxed{L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} a+2c & b+2d \\ -2a-c & -2b-d \end{bmatrix} + \begin{bmatrix} a-2c & 2a-c \\ b-2d & 2b-d \end{bmatrix} = \begin{bmatrix} 2a & 2a+b-c+2d \\ -2a+b-c-2d & -2d \end{bmatrix}$

$\boxed{E}$ $L(E) = \begin{bmatrix} 2 & 4 \\ -4 & -2 \end{bmatrix} \neq 0 \Rightarrow \boxed{E \notin \ker L}$, $\overset{\text{НПР}}{L}\left(\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}\right) = E \Rightarrow \boxed{E \in \operatorname{Im} L}$

$\boxed{F}$ $L(F) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{F \in \ker L}$, $L\left(\begin{bmatrix} 0 & 0 \\ 2024 & 0 \end{bmatrix}\right) = F \Rightarrow \boxed{F \in \operatorname{Im} L}$

$\boxed{G}$ $L(G) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \neq 0 \Rightarrow \boxed{G \notin \ker L}$, $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a & 2a+b-c+2d \\ -2a+b-c-2d & -2d \end{bmatrix} \neq G$

јер
$2a = 0 \Rightarrow a = 0$
$2a+b-c+2d = 0$
$-2a+b-c-2d = 1$ $\Big\} \Rightarrow$ $b-c=0$, $b-c=1$ ↯ $\Rightarrow$ нема решења $\Rightarrow \boxed{G \notin \operatorname{Im} L}$
$-2d = 0 \Rightarrow d = 0$

ⓥ $\boxed{\ker L = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \right\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{matrix} a=0 \\ d=0 \\ b=c \end{matrix} \right\} = \mathcal{L}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)}$

$L(E_1) = \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix}$, $L(E_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $L(E_3) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $L(E_4) = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}$

у станд. бази

$\begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -2 \end{bmatrix} \Rightarrow$ $\boxed{\operatorname{Im} L = \mathcal{L}\left(\begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}\right)}$

$\boxed{\rho(L) = 3, \quad \delta(L) = 1}$

Крамер аналогно, $L$ је пресликавање $-L$ из групе Лаплас ∎

$\boxed{4}$ ЛАПЛАС
$a^2 x + 2aY = 1$
$2aX + a^2 Y = -1$

$\Delta = a^4 - 4a^2 = a^2(a-2)(a+2)$, $\Delta_X = a^2 + 2a = a(a+2)$, $\Delta_Y = -a^2 - 2a = -a(a+2)$

① $\boxed{a \notin \{0, 2, -2\}} \Rightarrow \Delta \neq 0$
$\Rightarrow$ јединствено решење
$X = \frac{\Delta_X}{\Delta} = \frac{1}{a(a-2)}$
$Y = \frac{\Delta_Y}{\Delta} = \frac{-1}{a(a-2)}$

② $\boxed{a = 2}$
$\Rightarrow \Delta = 0$
$\Delta_X = 8 \neq 0$
$\Rightarrow$ нема решења

③ $\boxed{a = 0}$
$\Delta = \Delta_X = \Delta_Y = 0$
$0 = 1$
$0 = -1$
$\Rightarrow$ нема решења

④ $\boxed{a = -2} \Rightarrow \Delta = \Delta_X = \Delta_Y = 0$
$4X - 4Y = 1$
$-4X + 4Y = -1$ $\quad + \quad 0 = 0$
$(X, Y) = (\alpha, -\frac{1}{4} + \alpha)$
$\alpha \in \mathbb{R} \Rightarrow$ беск. решења

Крамер
$3bX + b^2 Y = -1$
$b^2 X + 3bY = 1$

$\Delta = 9b^2 - b^4 = b^2(3-b)(3+b)$, $\Delta_X = -3b - b^2 = -b(3+b)$, $\Delta_Y = 3b + b^2 = b(3+b)$

① $\boxed{b \notin \{0, \pm 3\}} \Rightarrow \Delta \neq 0$
$\Rightarrow$ јединствено решење
$X = \frac{\Delta_X}{\Delta} = \frac{-1}{b(3-b)}$
$Y = \frac{\Delta_Y}{\Delta} = \frac{1}{b(3-b)}$

② $\boxed{b = 3}$
$\Rightarrow \Delta = 0$
$\Delta_Y = 18 \neq 0$
$\Rightarrow$ нема решења

③ $\boxed{b = 0}$
$\Delta = \Delta_X = \Delta_Y = 0$
$0 = -1$
$0 = 1$
$\Rightarrow$ нема решења

④ $\boxed{b = -3} \Rightarrow \Delta = \Delta_X = \Delta_Y = 0$
$-9X + 9Y = -1$
$9X - 9Y = 1$
$(X, Y) = (\alpha, \alpha - \frac{1}{9}), \alpha \in \mathbb{R}$
$\Rightarrow$ бесконачно решења ∎

$\boxed{5}$ $L(e_1) = 2e_1 - e_2 + 2e_3$, $L(e_2) = e_1 + e_3$, $L(e_3) = -e_1 + 2e_2 + e_3$

**ЛАПЛАС**

$$\begin{aligned} \boxed{e_1} + 2e_2 + 3e_3 &= w_3 \quad \cdot 3,\ \cdot(-2) \\ -3e_1 - 5e_2 - 7e_3 &= w_2 \\ 2e_1 + 4e_2 + 5e_3 &= w_1 \end{aligned}$$

$$\begin{aligned} e_1 + 2e_2 + 3e_3 &= w_3 \\ e_2 + 2e_3 &= w_2 + 3w_3 \\ -e_3 &= w_1 - 2w_3 \end{aligned}$$

$$\boxed{\begin{aligned} e_1 &= -w_1 - 2w_2 - 3w_3 \\ e_2 &= 2w_1 + w_2 - w_3 \\ \Rightarrow e_3 &= -w_1 + 2w_3 \end{aligned}} \quad (*)$$

$$\begin{aligned} L(w_1) &= L(2e_1 + 4e_2 + 5e_3) \overset{\text{лин}}{=} 2L(e_1) + 4L(e_2) + 5L(e_3) \\ &= 2(2e_1 - e_2 + 2e_3) + 4(e_1 + e_3) + 5(-e_1 + 2e_2 + e_3) \\ &= 3e_1 + 8e_2 + 13e_3 \overset{(*)}{=} 0 \cdot w_1 + 2w_2 + 9w_3 \end{aligned}$$

$$\begin{aligned} L(w_2) &= L(-3e_1 - 5e_2 - 7e_3) = -3L(e_1) - 5L(e_2) - 7L(e_3) \\ &= -3(2e_1 - e_2 + 2e_3) - 5(e_1 + e_3) - 7(-e_1 + 2e_2 + e_3) \\ &= -4e_1 - 11e_2 - 18e_3 \overset{(*)}{=} 0 \cdot w_1 - 3w_2 - 13w_3 \end{aligned}$$

$$\begin{aligned} L(w_3) &= L(e_1 + 2e_2 + 3e_3) = L(e_1) + 2L(e_2) + 3L(e_3) \\ &= 2e_1 - e_2 + 2e_3 + 2e_1 + 2e_3 - 3e_1 + 6e_2 + 3e_3 \\ &= e_1 + 5e_2 + 7e_3 \overset{(*)}{=} 2w_1 + 3w_2 + 6w_3 \end{aligned}$$

$$\boxed{[L]_{S'} = \big[\,[Lw_1]_{S'}\ [Lw_2]_{S'}\ [Lw_3]_{S'}\,\big] = \begin{bmatrix} 0 & 0 & 2 \\ 2 & -3 & 3 \\ 9 & -13 & 6 \end{bmatrix}}$$

**КРАМЕР**

$$\begin{aligned} e_1 + 2e_2 + 3e_3 &= w_1 \\ -3e_1 - 5e_2 - 7e_3 &= w_3 \\ 2e_1 + 4e_2 + 5e_3 &= w_2 \end{aligned} \overset{\text{исто}}{\Longrightarrow} \boxed{\begin{aligned} e_1 &= -3w_1 - w_2 - 2w_3 \\ e_2 &= -w_1 + 2w_2 + w_3 \\ e_3 &= 2w_1 - w_2 \end{aligned}} \quad (\#)$$

$$L(w_1) = e_1 + 5e_2 + 7e_3 \overset{(\#)}{=} 6w_1 + 2w_2 + 3w_3$$

$$L(w_2) = 3e_1 + 8e_2 + 13e_3 = 9w_1 + 0 \cdot w_2 + 2w_3$$

$$L(w_3) = -4e_1 - 11e_2 - 18e_3 = -13w_1 + 0 \cdot w_2 - 3w_3$$

$$\Rightarrow \boxed{[L]_{S'} = \begin{bmatrix} 6 & 9 & -13 \\ 2 & 0 & 0 \\ 3 & 2 & -3 \end{bmatrix}}$$