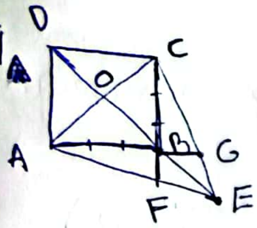
АГ - Г1 - СЕПТ 2 - 2024

1



Означимо са $P = P_{ABCD} = \|\vec{AB}\times\vec{AD}\|$, $\vec{OE} = \frac{3}{2}\vec{OB}$

$$P_{\triangle AEC} = \frac12\|\vec{AE}\times\vec{AC}\| = \frac12\|(\vec{AB}+\underbrace{\vec{BE}}_{1/2\,\vec{OB}})\times(\vec{AB}+\vec{BC})\|$$

$$= \frac12\|(\vec{AB}+\tfrac12(\vec{OA}+\vec{AB}))\times(\vec{AB}\times\vec{BC})\| = \frac12\|(\tfrac32\vec{AB}-\tfrac12\vec{AD})\times(\vec{AB}+\vec{AD})\|$$

$$= \frac12\|\tfrac32\vec{AB}\times\vec{AD}-\tfrac12\vec{AD}\times\vec{AB}\| = \frac12\|2\vec{AB}\times\vec{AD}\| = \boxed{P}$$

$O = AC\cap BD$

$\triangle BOC$
$E\in BO,\ F\in BC,\ A\in CO$ колинеарне } Менелај $\Rightarrow$ $\underbrace{\frac{\vec{BE}}{\vec{EO}}}_{=-1/2}\cdot\underbrace{\frac{\vec{OA}}{\vec{AC}}}_{-1/2}\cdot\frac{\vec{CF}}{\vec{FB}} = -1 \Rightarrow \vec{CF} = -4\vec{FB}$

$\Rightarrow \vec{BF} = -\frac13\vec{BC} = -\frac13\vec{AD}$

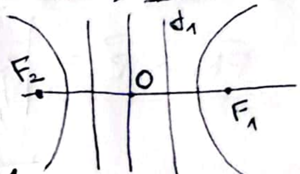
$\triangle AOB$
$G\in AB,\ E\in BO,\ C\in OA$ колинеарне } Менелај $\Rightarrow$ $\frac{\vec{AG}}{\vec{GB}}\cdot\underbrace{\frac{\vec{BE}}{\vec{EO}}}_{-1/2}\cdot\underbrace{\frac{\vec{OC}}{\vec{CA}}}_{-1/2} = -1 \Rightarrow \vec{AG} = -4\vec{GB}$

$\Rightarrow \vec{BG} = \frac13\vec{AB}$

$$P_{\triangle BFG} = \frac12\|\vec{BF}\times\vec{BG}\| = \frac12\|-\tfrac13\vec{AD}\times\tfrac13\vec{AB}\| = \frac12\cdot\frac19\|\vec{AB}\times\vec{AD}\| = \boxed{\frac{P}{18}}$$ ■

2

$f_d(x,y) = x+2y+2$, $f_d(F_1) = f_d(2,0) = 4$, $f_d(F_2) = f_d(0,-4) = -6$ $\Rightarrow$ F_1 и F_2 са супр. страна d $\Rightarrow$ хипербола , F_1 и d пар



центар : $O = S(F_1F_2) \Rightarrow O(1,-2)$

ексцентр: $c = d(O,F_1) = \sqrt{(-1)^2+(-2)^2} = \sqrt5 > 1$

$\frac{a^2}{c} = d(O,d) = \frac{|1-4+2|}{\sqrt{1+4}} = \frac{1}{\sqrt5} \Rightarrow a^2 = c\cdot\frac{1}{\sqrt5} = 1 \Rightarrow a = 1$

$e = \frac{c}{a} = \frac{\sqrt5}{1} = \boxed{\sqrt5}$ Нека је $M(x,y)\in\chi$ произв. $\Rightarrow \frac{d(M,F_1)}{d(M,d)} = e$

$\Rightarrow \chi: \dfrac{\sqrt{(x-2)^2+y^2}}{\frac{|x+2y+2|}{\sqrt5}} = \sqrt5 \Rightarrow \boxed{\chi:\ (x-2)^2+y^2 = (x+2y+2)^2}$

хипербола ■

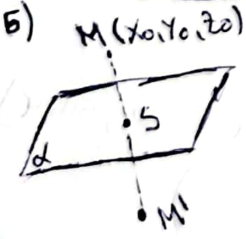
3

$\alpha: x-2y+z-3=0$, $\beta: y+2z=0$, $P(0,2,-1)$

а) p т.д. $P\in p$, $p\parallel\alpha \Rightarrow \vec p\perp\vec n_\alpha = (1,-2,1)$
$p\subset\beta \Rightarrow \vec p\perp\vec n_\beta = (0,1,2)$ $\Rightarrow \vec p = \vec n_\alpha\times\vec n_\beta = \begin{vmatrix} i & j & k\\ 1 & -2 & 1\\ 0 & 1 & 2\end{vmatrix} = (-5,-2,1)$

$\Rightarrow \boxed{p: \frac{x}{5} = \frac{y-2}{2} = \frac{z+1}{-1}}$

б) $M(x_0,y_0,z_0)$



1) n т.д. $M\in n$, $n\perp\alpha \Rightarrow n: \begin{cases} x = t+x_0\\ y = -2t+y_0\\ z = t+z_0\end{cases}$, $t\in\mathbb{R}$

2) $S = n\cap\alpha$: $t+x_0+4t-2y_0+t+z_0-3 = 0 \Rightarrow t = \frac{3-x_0+2y_0-z_0}{6}$

$\Rightarrow S\left(\frac{5x_0+2y_0-z_0+3}{6}, \frac{x_0+y_0+z_0-3}{3}, \frac{-x_0+2y_0+5z_0+3}{6}\right)$

3) $[M'] = 2[S]-[M]$ $\Rightarrow \boxed{S_\alpha: \begin{bmatrix} x'\\ y'\\ z'\end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & -1/3\\ 2/3 & -1/3 & 2/3\\ -1/3 & 2/3 & 2/3\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} + \begin{bmatrix} 1\\ -2\\ 1\end{bmatrix}}$

$S_\alpha(P) = P'(\frac83, \frac{-10}{3}, \frac53)$
$\Rightarrow S_\alpha(p) = p' = P'Q'$, $\vec{P'Q'} = (-5,-2,1)$

$Q(-5,0,0)\in p$
$\Rightarrow S_\alpha(Q) = (-\frac73, \frac{-16}{3}, \frac83)Q'$ $\Rightarrow \boxed{p': \frac{x-\frac83}{-5} = \frac{y+\frac{10}{3}}{-2} = \frac{y-5/3}{1}}$

$p\parallel\alpha \Rightarrow p'\parallel p$ па $\vec{p'} = \vec p$ (то. могли смо без Q) ■

4

$$A=\begin{bmatrix}3&2&4&0\\2&0&2&0\\4&2&3&-3/2\\0&0&-3/2&0\end{bmatrix} \Rightarrow B=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix} \qquad \varphi_B(\lambda)=\begin{vmatrix}3-\lambda&2&4\\2&-\lambda&2\\4&2&3-\lambda\end{vmatrix}$$

$$\dots = -(\lambda+1)^2(\lambda-8)$$

$$\lambda_1=\lambda_2=-1 \xRightarrow{-b=2a+2c} \dots \Rightarrow f_1=\begin{bmatrix}1/\sqrt5\\-2/\sqrt5\\0\end{bmatrix} \quad (*)\ f_2=\begin{bmatrix}4/3\sqrt5\\2/3\sqrt5\\-5/3\sqrt5\end{bmatrix},\ \lambda_3=8 \Rightarrow f_3=\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1/\sqrt5&4/3\sqrt5&2/3\\-2/\sqrt5&2/3\sqrt5&1/3\\0&-5/3\sqrt5&2/3\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\end{bmatrix}} \Rightarrow A'=R'^TAR'=\dots=\begin{bmatrix}-1&0&0&0\\0&-1&0&\frac{\sqrt5}{2}\\0&0&8&-1\\0&\sqrt5/2&-1&0\end{bmatrix}$$

(НА КРАЈУ ЛИСТА)

(*) МОРА $f_2 \perp f_1 \Rightarrow 2a+b+2c=0$ и $a-2b=0 \Rightarrow f_2=(4,2,-5)$

$$\Rightarrow S': -x'^2-y'^2+8z'^2+\sqrt5 y'-2z'=0$$

$$S': -x'^2-\underbrace{\left(y'-\frac{\sqrt5}{2}\right)}_{y''}{}^2+8\underbrace{\left(z'-\frac18\right)}_{z''}{}^2+\frac54-\frac18=0 \quad /\cdot(-1)$$

$$\Rightarrow \boxed{S'': x''^2+y''^2-8z''^2=9/8} \Rightarrow \boxed{\text{ЈЕДНОГРАНИ ХИПЕРБОЛОИД}}$$

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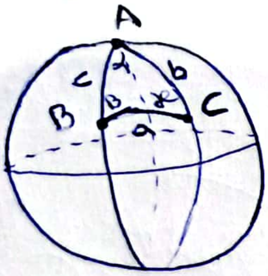
5 $A(0,0,1)$, $B\left(\frac{\sqrt3}{2},0,\frac12\right)$, $\widehat{AB}=\widehat{AC}$, $\measuredangle \widehat{BAC}=\pi/2$

$x=R\cos\theta\cos\varphi$
$y=R\sin\theta\cos\varphi$, $R=1$ $\Rightarrow$ $A: \varphi_A=\frac{\pi}{2}$, θ_A недеф.
$z=R\sin\varphi$ $\qquad B: \varphi_B=\frac{\pi}{6}$, $\theta_B=0$



$\widehat{AB}=\varphi_A-\varphi_B=\frac{\pi}{3}=\widehat{AC} \Rightarrow \varphi_C=\frac{\pi}{6}$

$\measuredangle\widehat{BAC}=\frac{\pi}{2} \Rightarrow \theta_C=\pm\frac{\pi}{2} \Rightarrow$ нека је $\boxed{C: \begin{matrix}\varphi_C=\pi/6\\ \theta_C=\pi/2\end{matrix}}$

$$\Rightarrow A(0,0,1),\ B\left(\frac{\sqrt3}{2},0,\frac12\right),\ C\left(0,\frac{\sqrt3}{2},\frac12\right)$$

$\cos a=\cos b\cos c+\sin b\sin c\,\cancel{\cos\alpha}^{\,0}=\frac14 \Rightarrow \boxed{a=\arccos\frac14,\ b=c=\frac{\pi}{3}}$

$\alpha=\frac{\pi}{2}$, $\beta=\gamma$ јер $b=c$

$\sin a=\sqrt{1-\cos^2 a}=\frac{\sqrt{15}}{4}$

$$\cos c=\cos a\cos b+\sin a\sin b\cos\gamma$$

$$\frac12=\frac14\cdot\frac12+\frac{\sqrt{15}}{4}\cdot\frac{\sqrt3}{2}\cos\gamma$$

$$\frac38=\frac{3\sqrt5}{8}\cos\gamma \Rightarrow \cos\gamma=\frac{1}{\sqrt5}=\cos\beta$$

$P=\alpha+\beta+\gamma-\pi=\boxed{2\arccos\frac{1}{\sqrt5}-\frac{\pi}{2}} \qquad \Rightarrow \boxed{\beta=\gamma=\arccos\frac{1}{\sqrt5}}$

Појашњење за **4** (*):

$$\lambda_1=\lambda_2=-1 \Rightarrow \begin{bmatrix}4&2&4\\2&1&2\\4&2&4\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=0$$

$\Rightarrow 2a+b+2c=0 \Rightarrow b=-2a-2c$

$$V=\begin{bmatrix}a\\-2a-2c\\c\end{bmatrix}=\mathcal{L}\left(\overset{f_1}{\begin{bmatrix}1\\-2\\0\end{bmatrix}},\overset{f_2}{\begin{bmatrix}0\\-2\\1\end{bmatrix}}\right)$$

Како f_1 и f_2 нису ортогонални, уместо f_2 узимамо други с.в. $\perp f_1$

$\tilde f_2=(a,b,c)$ т.д. $2a+b+2c=0$ (с.в.)

$f_1\perp\tilde f_2=0 \Rightarrow a-2b=0 \Rightarrow a=2b$

$\Rightarrow c=-\frac52 b$

$$\Rightarrow \tilde f_2=\begin{bmatrix}2b\\b\\-5/2\,b\end{bmatrix}\sim\begin{bmatrix}4\\2\\-5\end{bmatrix} \text{ па нормирано}$$