АГ - Г1 - ЈУН 2 - 2024

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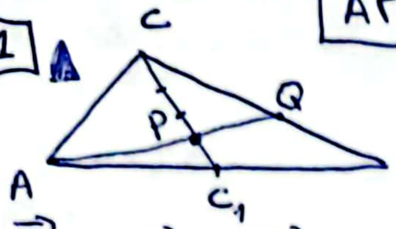

I начин $\vec{AP}=\alpha\vec{AQ}$, $\vec{BQ}=\beta\vec{BC}$

$$\vec{AP}=\alpha\vec{AQ}=\alpha(\vec{AB}+\vec{BQ})=\alpha(\vec{AB}+\beta(\vec{BA}+\vec{AC}))=$$
$$=(\alpha-\alpha\beta)\vec{AB}+\alpha\beta\vec{AC}$$

$$\vec{AP}=\vec{AC_1}+\vec{C_1P}=\frac{1}{2}\vec{AB}+\frac{1}{4}\vec{C_1C}=\frac{1}{2}\vec{AB}+\frac{1}{4}(\underbrace{\vec{C_1A}}_{-\frac{1}{2}\vec{AB}}+\vec{AC})=\frac{3}{8}\vec{AB}+\frac{1}{4}\vec{AC}$$

$\Rightarrow (\alpha-\alpha\beta-\frac{3}{8})\vec{AB}+(\alpha\beta-\frac{1}{4})\vec{AC}=0$, $\vec{AB}$ и $\vec{AC}$ линеарно независни

$\Rightarrow \alpha-\alpha\beta=3/8$, $\alpha\beta=1/4$ $\Rightarrow$ $\alpha=5/8$, $\beta=2/5$ $\Rightarrow \vec{AP}=\frac{5}{8}\vec{AQ}$, $\vec{BQ}=\frac{2}{5}\vec{BC}$ $\Rightarrow$ $\boxed{BQ:BC=2:5 \quad AP:AQ=5:8}$

II начин ΔBCC_1

$Q\in BC$, $P\in CC_1$, $A\in C_1B$ колинеарне $\overset{\text{Менелај}}{\Longleftrightarrow}$ $\frac{\vec{BQ}}{\vec{QC}}\cdot\underbrace{\frac{\vec{CP}}{\vec{PC_1}}}_{=3}\cdot\underbrace{\frac{\vec{C_1A}}{\vec{AB}}}_{=-1/2}=-1 \Rightarrow \boxed{\frac{\vec{BQ}}{\vec{QC}}=\frac{2}{3}} \Rightarrow \boxed{\frac{BQ}{BC}=\frac{2}{5}}$

ΔAQB

$P\in AQ$, $C\in QB$, $C_1\in BA$ колинеарне $\overset{\text{Менелај}}{\Longleftrightarrow}$ $\frac{\vec{AP}}{\vec{PQ}}\cdot\underbrace{\frac{\vec{QC}}{\vec{CB}}}_{=-3/5}\cdot\underbrace{\frac{\vec{BC_1}}{\vec{C_1A}}}_{=1}=-1 \Rightarrow \frac{\vec{AP}}{\vec{PQ}}=\frac{5}{3} \Rightarrow \boxed{\frac{AP}{AQ}=\frac{5}{8}}$

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2 а) $p: \begin{cases} x=1 \\ y=t+1 \\ z=-t-2 \end{cases}, t\in\mathbb{R}$, $q: \begin{cases} x=s-2 \\ y=s+1 \\ z=1 \end{cases}, s\in\mathbb{R}$ $\Rightarrow$ $P(1,t+1,-t-2)\in p$ произв, $Q(s-2,s+1,1)\in q$ произв.

$\Rightarrow \vec{PQ}=(s-3, s-t, t+3)$, $s,t\in\mathbb{R}$, $\vec{p}=(0,1,-1)$, $\vec{q}=(1,1,0)$

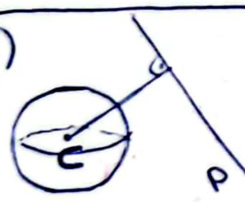
1) $\vec{PQ}\circ\vec{p}=0 \Rightarrow s-t-t-3=0 \Rightarrow s-2t=3 \;/\cdot(-2)$ $\quad 3t=-3 \Rightarrow \boxed{t=-1}$

2) $\vec{PQ}\circ\vec{q}=0 \Rightarrow s-3+s-t=0 \Rightarrow 2s-t=3$ $\quad \boxed{s=1}$

$\Rightarrow P(1,0,-1)$, $Q(-1,2,1)$, $\vec{PQ}=(-2,2,2)\parallel(-1,1,1)$

$\boxed{\text{ЗАЈЕДНИЧКА НОРМАЛА} \quad n=PQ: \frac{x-1}{-1}=\frac{y}{1}=\frac{z+1}{1}}$ $\quad d(p,q)=\|\vec{PQ}\|=\sqrt{2^2+2^2+2^2}=\boxed{2\sqrt{3}}$

б)



Најближа тачка је подножје нормале из центра сфере C на праву p, $C(7,5,0)$

$P(1,t+1,-t-2)\in p$ произв $\Rightarrow \vec{CP}=(-6,t-4,-t-2)$

$\vec{CP}\circ\vec{p}=0 \Rightarrow t-4+t+2=0 \Rightarrow t=1$

$\Rightarrow$ тачка $\boxed{A(1,2,-3)\in p}$ је најближа $\sigma: (x-7)^2+(y-5)^2+z^2=16$

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3 Нека је $F(x_0,y_0)$ жижа $\mathcal{P}$, $d: 2x-y+2+2\sqrt{5}=0$, $\sigma: x+2y-3=0$

1) $F(x_0,y_0)\in\sigma \Rightarrow x_0=3-2y_0 \Rightarrow F(3-2y_0,y_0)$, $A(-1,0)$

2) $d(A,F)=d(A,d) \Rightarrow \sqrt{(4-2y_0)^2+y_0^2}=\frac{|\cancel{-2}-0\cancel{+2}+2\sqrt{5}|}{\sqrt{5}}=2 \;/^2$

$\Rightarrow 16-16y_0+4y_0^2+y_0^2=4$

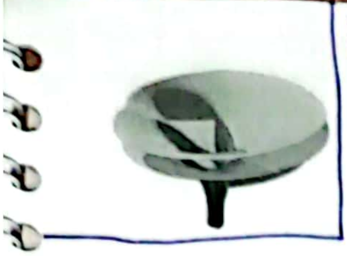
$5y_0^2-16y_0+12=0 \Rightarrow y_0=\frac{16\pm\sqrt{256-240}}{10}=\frac{16\pm4}{10} \nearrow 2, \searrow \frac{6}{5}$

$\Rightarrow F_1(-1,2)$, $F_2(3/5,6/5)$ $\Rightarrow$ има два решења

I $M(x,y)\in\mathcal{P}_1$ произв. $\Rightarrow \mathcal{P}_1: d(M,F_1)=d(M,d)$

$\sqrt{(x+1)^2+(y-2)^2}=\frac{|2x-y+2+2\sqrt{5}|}{\sqrt{5}} \;/^2 \Rightarrow \boxed{\mathcal{P}_1: (x+1)^2+(y-2)^2=\frac{(2x-y+2+2\sqrt{5})^2}{5}}$

II $\mathcal{P}_2: d(M,F_2)=d(M,d) \Rightarrow \boxed{\mathcal{P}_2: \left(x-\frac{3}{5}\right)^2+\left(y-\frac{6}{5}\right)^2=\frac{(2x-y+2+2\sqrt{5})^2}{5}}$



4 $P: y=0, z=0, \varphi = \pi/3, \ \alpha: x=0$

$$R_{P,\varphi}: \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\vec{n}_\alpha = (1,0,0)^T$ јединични, α пролази кроз коорд. поч

$$\Rightarrow [S_\alpha] = I_3 - 2\vec{n}_\alpha \cdot \vec{n}_\alpha^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow S_\alpha: \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow [R_{P,\varphi} \circ S_\alpha] = [R_{P,\varphi}][S_\alpha] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow \boxed{R_{P,\varphi} \circ S_\alpha: \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

овде је $R_{P,\varphi} \circ S_\alpha = S_\alpha \circ R_{P,\varphi}$ па је свеједно којим редоследом

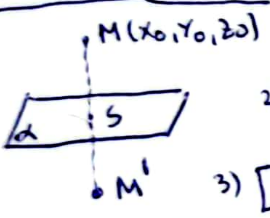
* пошто је у питању изометрија, сфера се слика у сферу истог полупр, а центар у центар

$\sigma: x^2 + (y-1)^2 + z^2 = 1 \Rightarrow R = 1, \ C(0,1,0)$

$$C' = R_{P,\varphi} \circ S_\alpha (C) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix} \Rightarrow C'\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \sigma \longmapsto \boxed{\sigma': x^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{\sqrt{3}}{2}\right)^2 = 1}$$

II начин за S_α:



1) h т.д. $M \in h$, $h \perp \alpha \Rightarrow h: \begin{cases} x = t + x_0 \\ y = y_0 \\ z = z_0 \end{cases}, \ t \in \mathbb{R}$

$\vec{n} = \vec{n}_\alpha = (1,0,0)$

2) $S = h \cap \alpha \Rightarrow t + x_0 = 0 \Rightarrow t = -x_0 \Rightarrow S(0, y_0, z_0)$

3) $[M'] = 2[S] - [M] = (0, 2y_0, 2z_0) - (x_0, y_0, z_0) = (-x_0, y_0, z_0)$

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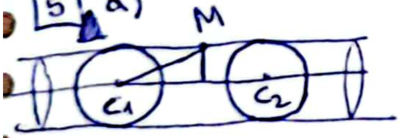
5 а) $C_1(0,0,0), \ C_2(1,1,-2), \ R_1 = R_2 = 1$



оса цилиндра је $p: C_1C_2: \frac{x}{1} = \frac{y}{1} = \frac{z}{-2}$

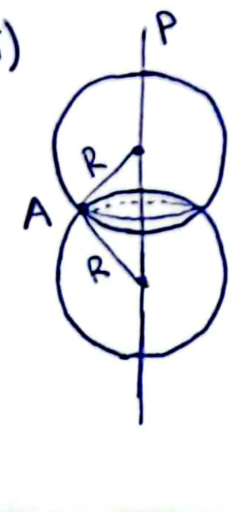
$\Rightarrow \mathcal{C}$ кружни цилиндар са осом CC_1 и $R = 1$

$M(x,y,z) \in \mathcal{C}$ произв. $\Rightarrow \mathcal{C}: d(M, C_1C_2) = R = 1$

$$\mathcal{C}: \frac{\|\vec{C_1M} \times \vec{C_1C_2}\|}{\|\vec{C_1C_2}\|} = 1 \ /^2 \Rightarrow \boxed{\mathcal{C}: (-2y-z)^2 + (2x+z)^2 + (x-y)^2 = 6}$$

$$\vec{C_1M} \times \vec{C_1C_2} = \begin{vmatrix} i & j & k \\ x & y & z \\ 1 & 1 & -2 \end{vmatrix} = (-2y-z, 2x+z, x-y)$$

б)



$A(1,1,1)$

$C(t,t,-2t) \in p$ произв $\Rightarrow d(C,A) = 3$

$$\sqrt{(t-1)^2 + (t-1)^2 + (-2t-1)^2} = 3 \ /^2$$

$$t^2 \cancel{-2t} + 1 + t^2 \cancel{-2t} + 1 + 4t^2 \cancel{+4t} + 1 = 9$$

$$6t^2 = 6 \Rightarrow t_1 = -1 \quad t_2 = 1$$

$C_1(1,1,-2) \quad C_2(-1,-1,2)$

$$\boxed{\begin{aligned} S_1 &: (x-1)^2 + (y-1)^2 + (z+2)^2 = 9 \\ S_2 &: (x+1)^2 + (y+1)^2 + (z-2)^2 = 9 \end{aligned}}$$