$\boxed{\text{ЛААГ - септ1 - 2023}}$

1] већ виђено у пар рокова

2] $L(X) = XA - 2tr(X)\cdot B$, $A = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$, $L : M_2(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$

L линеарно?

1) $L(X+Y) = (X+Y)A - 2tr(X+Y)\cdot B = XA + YA - 2tr(X)B - 2tr(Y)B$

$= L(X) + L(Y)$, користили смо $tr(X+Y) = tr(X) + tr(Y)$

2) $L(\alpha X) = (\alpha X)A - 2tr(\alpha X)B = \alpha XA - 2\alpha\, tr(X)B = \alpha L(X)$

јер је $tr(\alpha X) = \alpha\, tr(X)$ $\overset{1),2)}{\Longrightarrow}$ L је линеарно

$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} - 2(a+d)\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 2a & -2a & 4a \\ 2c & -2c & 4c \end{bmatrix} - \begin{bmatrix} 2a+2d & -2a-2d & 4a+4d \\ -2a-2d & 2a+2d & -4a-4d \end{bmatrix}$

$$\boxed{L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -2d & 2d & -4d \\ 2(a+c+d) & -2(a+c+d) & 4(a+c+d) \end{bmatrix}}$$

* $KerL = \{X \in M_2(\mathbb{R}) \mid L(X) = 0\} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, \begin{matrix} d=0 \\ a+c+d=0 \end{matrix}\right\} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, \begin{matrix} d=0 \\ c=-a \end{matrix}\right\}$

$= \left\{\begin{bmatrix} a & b \\ -a & 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\} = \mathcal{L}\left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \Rightarrow \boxed{\delta(L) = 2}$

* $ImL = \mathcal{L}(L(E_1), L(E_2), L(E_3), L(E_4))$

$L(E_1) = L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$, $L(E_2) = L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$L(E_3) = L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$, $L(E_4) = L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -2 & 2 & -4 \\ 2 & -2 & 4 \end{bmatrix}$

(одавде се може видети $E_1 - E_3 \in KerL$, $E_2 \in KerL$)

$\Rightarrow ImL = \mathcal{L}\left(\begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 2 & -4 \\ 2 & -2 & 4 \end{bmatrix}\right) \Rightarrow \boxed{\rho(L) = 2}$

3] $\varphi_A(\lambda) = \begin{vmatrix} 2023-\lambda & 2022 & 2023 \\ 0 & 2022-\lambda & 2022 \\ 0 & 0 & 2023-\lambda \end{vmatrix} = (2022-\lambda)(2023-\lambda)^2$

$\Rightarrow \mu_A(\lambda) = (\lambda-2022)(\lambda-2023)$ или $\mu_A(\lambda) = (\lambda-2022)(\lambda-2023)^2$

$(A-2022E)(A-2023E) = \begin{bmatrix} 1 & 2022 & 2023 \\ 0 & 0 & 2022 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2022 & 2023 \\ 0 & -1 & 2022 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2023+2022^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \boxed{\mu_A(\lambda) = (\lambda-2022)(\lambda-2023)^2}$ $\Rightarrow$ A није дијаг. типа јер μ_A има вишестр. нуле

* $\lambda_1 = 2022$

$\begin{bmatrix} 1 & 2022 & 2023 \\ 0 & 0 & 2022 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$a + 2022b + 2023c = 0$

$c = 0$

$\boxed{v_1 = b(-2022, 1, 0)}$

* $\lambda_2 = 2023$

$\begin{bmatrix} 0 & 2022 & 2023 \\ 0 & -1 & 2022 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$2022b + 2023c = 0$

$-b + 2022c = 0$

$\Rightarrow b = 0, c = 0$

$\boxed{v_2 = a(1, 0, 0)}$

$\Rightarrow$ A није дијаг. типа јер је реда 3 а има само два сопствена вектора

$\boxed{4}$ $\langle (x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = 3x_1y_1 + 2x_2y_2 + 2x_3y_3 + 2x_3y_1 + 2x_1y_3$

а) $\langle , \rangle$ скаларни ? $\Leftrightarrow$ линеарност, комутат, поз. дефинитност ... лако

б) $v=(1,1,1) = v_1+v_2 = \alpha f_1 + \beta f_2 + v_2$, $f_1=(1,1,0)$, $f_2=(-1,0,1)$, $v_1\in U$, $v_2 \in U^{\perp}$

$\langle v,f_1\rangle = 3+2+0+2+0 = 7$ $\qquad$ $\langle f_1,f_1\rangle = 3+2+0+0+0=5$

$\langle v,f_2\rangle = -3+0+2-2+2 = -1$ $\qquad$ $\langle f_1,f_2\rangle = -3+0+0+0+2 = -1$

$\langle f_2,f_2\rangle = 3+0+2-2-2 = 1$

$\Rightarrow 7 = 5\alpha - \beta \Rightarrow 4\alpha = 6$

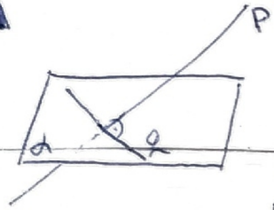
$\underline{-1 = -\alpha+\beta}$ $\quad \alpha = \frac{3}{2}$ $\quad \beta = \frac{1}{2}$

$v_1 = \frac{3}{2}(1,1,0)+\frac{1}{2}(-1,0,1) = (1,\frac{3}{2},\frac{1}{2})$

$v_2 = v - v_1 = (0,-\frac{1}{2},\frac{1}{2})$

$d(v,U) = \|v-v_1\| = \|v_2\| = \sqrt{0+\frac{1}{2}+\frac{1}{2}+0+0} = \sqrt{1} = \boxed{1}$

$\boxed{5}$ $p: \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$, $\alpha: 2x-y+z-6=0$



$p\cap\alpha$: $\quad p: \begin{array}{l} x = t+1 \\ y = 2t+2, t\in\mathbb{R} \\ z = 3t+3 \end{array} \Rightarrow 2t+2-2t-2+3t+3-6=0 \Rightarrow t=1$

$\Rightarrow p\cap\alpha = A(2,4,6)\in q$

$q\subset\alpha \Rightarrow \vec{q}\perp\vec{n}_\alpha = (2,-1,1)$

$q\perp p \Rightarrow \vec{q}\perp\vec{p} = (1,2,3)$ $\Rightarrow \vec{q} = \vec{n}_\alpha\times\vec{p} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 2 & 3\end{vmatrix} = (-5,-5,5) \parallel (1,1,-1)$

$\Rightarrow \boxed{q: \frac{x-2}{1} = \frac{y-4}{1} = \frac{z-6}{-1}}$

$\boxed{6}$ $U,W\leq V$, $\dim V = 6$, $\dim U = \dim W = 4$

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* $\dim U\cap W = \underset{4}{\dim U} + \underset{4}{\dim W} - \underset{\leq 6}{\dim U+W} \geq 2 \Rightarrow \boxed{\dim U\cap W \geq 2}$
* $U\cap W \leq U, W \Rightarrow \dim U\cap W \leq \dim U = 4$

$\Rightarrow \boxed{2\leq \dim U\cap W \leq 4}$

$\boxed{\text{I}}$ $V=\mathbb{R}^6$

$U = \mathcal{L}(e_1,e_2,e_3,e_4)$

$W = \mathcal{L}(e_1,e_2,e_3,e_4)$

$\Rightarrow U\cap W = U = W = U+W$

$\boxed{\begin{array}{l}\dim U\cap W = 4 \\ \dim U+W = 4\end{array}}$

$\boxed{\text{II}}$ $V=\mathbb{R}^6$

$U = \mathcal{L}(e_1,e_2,e_3,e_4)$

$W = \mathcal{L}(e_1,e_2,e_3,e_5)$

$U\cap W = \mathcal{L}(e_1,e_2,e_3)$

$U+W = \mathcal{L}(e_1,e_2,e_3,e_4,e_5)$

$\boxed{\begin{array}{l}\dim U\cap W = 3 \\ \dim U+W = 5\end{array}}$

$\boxed{\text{III}}$ $V=\mathbb{R}^6$

$U = \mathcal{L}(e_1,e_2,e_3,e_4)$

$W = \mathcal{L}(e_1,e_2,e_5,e_6)$

$U\cap W = \mathcal{L}(e_1,e_2)$

$U+W = \mathcal{L}(e_1,e_2,e_3,e_4,e_5,e_6) = \mathbb{R}^6$

$\boxed{\begin{array}{l}\dim U\cap W = 2 \\ \dim U+W = 6\end{array}}$