$\boxed{1}$ већ виђено $\qquad$ $\boxed{\text{ЛААГ - ЈУН 1 - 2024}}$ $\qquad$ $a^3 - a - 2a + 2 = a(a^2-1) - 2(a-1) = (a-1)(a(a+1)-2) = (a-1)(a^2+a-2)$

$\boxed{2}$

$$\Delta = \begin{vmatrix} a & 1 & b \\ 1 & a & b \\ 1 & 1 & ab \end{vmatrix}\begin{matrix} a & 1 \\ 1 & a \\ 1 & 1 \end{matrix} = \begin{matrix} a^3b + b + b \\ -ab - ab - ab \end{matrix} = b(a^3 - 3a + 2) = \boxed{b(a-1)^2(a+2)}$$

$$\Delta_x = \begin{vmatrix} 1 & 1 & b \\ 1 & a & b \\ b & 1 & ab \end{vmatrix}\begin{matrix} 1 & 1 \\ 1 & a \\ b & 1 \end{matrix} = \begin{matrix} a^2b + b^2 + \cancel{b} \\ -ab^2 - \cancel{b} - ab \end{matrix} = a^2b - ab + b^2 - ab^2 = ab(a-1) + b^2(1-a)$$
$$= (a-1)(ab - b^2) = \boxed{b(a-1)(a-b)}$$

$$\Delta_y = \begin{vmatrix} a & 1 & b \\ 1 & 1 & b \\ 1 & b & ab \end{vmatrix}\begin{matrix} a & 1 \\ 1 & 1 \\ 1 & b \end{matrix} = \begin{matrix} a^2b + \cancel{b} + b^2 \\ -\cancel{b} - ab^2 - ab \end{matrix} = \boxed{b(a-1)(a-b)}$$

$$\Delta_z = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & b \end{vmatrix}\begin{matrix} a & 1 \\ 1 & a \\ 1 & 1 \end{matrix} = \begin{matrix} a^2b + 1 + 1 \\ -a - a - b \end{matrix} = b(a^2-1) - 2(a-1) = (a-1)(b(a+1)-2)$$
$$= \boxed{(a-1)(ab+b-2)}$$

$\boxed{\text{I}}$ $\boxed{b \neq 0,\ a \neq 1,\ a \neq -2} \Rightarrow \Delta \neq 0 \Rightarrow$ јединствено решење

$$\boxed{x = \frac{\Delta x}{\Delta} = \frac{a-b}{(a-1)(a+2)},\quad y = \frac{\Delta y}{\Delta} = \frac{a-b}{(a-1)(a+2)},\quad z = \frac{\Delta z}{\Delta} = \frac{ab+b-2}{b(a-1)(a+2)}}$$

$\boxed{\text{II}}$ $b = 0 \Rightarrow \Delta = \Delta_x = \Delta_y = 0$, $\Delta_z = -2(a-1)$

II.1 $a \neq 1 \Rightarrow \Delta_z \neq 0 \Rightarrow$ систем нема решења

II.2 $a = 1 \Rightarrow$ све детерминанте су нула

$$\Rightarrow \begin{matrix} x + y = 1 \\ x + y = 1 \\ x + y = 0 \end{matrix} \Rightarrow \text{систем нема решења}$$

$\boxed{\begin{matrix} b = 0 \\ \text{нема решења} \end{matrix}}$

$\boxed{\text{III}}$ $a = 1 \Rightarrow \Delta = \Delta_x = \Delta_y = \Delta_z = 0 \Rightarrow$ не може Крамер

$$\Rightarrow \begin{matrix} x + y + bz = 1 & / \cdot (-1) \\ x + y + bz = 1 & \\ x + y + bz = b & \end{matrix} \qquad \begin{matrix} x + y + bz = 1 \\ 0 = b - 1 \end{matrix} \Rightarrow$$

$\boxed{\text{III.1}}$ $a = 1,\ b \neq 1$ $\Rightarrow$ нема решења

$\boxed{\text{III.2}}$ $a = 1,\ b = 1 \Rightarrow x + y + z = 1 \Rightarrow$

$(x, y, z) = (\alpha, \beta, 1 - \alpha - \beta)$, $\alpha, \beta \in \mathbb{R} \Rightarrow$ бесконачно решења

$\boxed{\text{IV}}$ $a = -2 \Rightarrow \Delta = 0$

$\Delta_x = -3b(-2-b) = 3b(b+2)$, $\Delta_y = 3b(b+2)$, $\Delta_z = 3(b+2)$

IV.1 $a = -2,\ b \neq -2 \Rightarrow \Delta_z \neq 0 \Rightarrow$ систем нема решења

IV.2 $a = -2,\ b = -2 \Rightarrow \Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$$\begin{matrix} -2x + y - 2z = 1 \\ x - 2y - 2z = 1 \\ x + y + 4z = -2 & / \cdot (-1) \ / \cdot 2 \end{matrix} \Rightarrow \begin{matrix} x + y + 4z = -2 \\ -3y - 6z = 3 \\ 3y + 6z = -3 \end{matrix} \Rightarrow \begin{matrix} x + y + 4z = -2 \\ y + 2z = 1 \end{matrix}$$

$$\boxed{z = \alpha,\ y = 1 - 2\alpha = x,\ \alpha \in \mathbb{R}}$$

$\boxed{\text{Закључак}}$

1) $b \neq 0,\ a \neq 1,\ a \neq -2 \Rightarrow$ јединствено решење

2) $a = 1,\ b = 1 \Rightarrow$ бесконачно решења $(x, y, z) = (\alpha, \beta, 1 - \alpha - \beta)$, $\alpha, \beta \in \mathbb{R}$

3) $a = -2,\ b = -2 \Rightarrow$ бесконачно реш. $(x, y, z) = (1 - 2\alpha, 1 - 2\alpha, \alpha)$, $\alpha \in \mathbb{R}$

4) сви остали случајеви $\Rightarrow$ нема решења

$\boxed{3}$ $L : M_2(\mathbb{R}) \to \mathbb{R}^2$, $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+2b+3c,\ b+c-2d)$.

а) $L(E_1) = L\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = (6,0)_K = (2,-4)_F$ $\quad + \ (a,b) = x \overset{(1,2)}{f_1} + y \overset{(-1,1)}{f_2}$

$L(E_2) = L\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = (5,0)_K = (5/3, -10/3)_F$ $\quad (a,b) = (x-y,\ 2x+y)$

$L(E_3) = L\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = (3,-1)_K = (2/3, -7/3)_F$ $\quad \begin{matrix} x-y=a \\ 2x+y=b \end{matrix} \Rightarrow \begin{matrix} x = \frac{1}{3}(a+b) \\ y = -\frac{2}{3}a + \frac{1}{3}b \end{matrix}$

$L(E_4) = L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (0,-2)_K = (-2/3, -2/3)_F$ $\quad \underset{\text{канонска}}{(a,b)_K} = \left(\frac{1}{3}a + \frac{1}{3}b,\ -\frac{2}{3}a + \frac{1}{3}b\right)_F$

$$\Rightarrow \boxed{[L]_E^F = \begin{bmatrix} 2 & 5/3 & 2/3 & -2/3 \\ -4 & -10/3 & -7/3 & -2/3 \end{bmatrix}}$$

б) $M_2(\mathbb{R}) = \mathcal{L}(E_1, E_2, E_3, E_4) \Rightarrow \operatorname{Im} L = \mathcal{L}(L(E_1), L(E_2), L(E_3), L(E_4))$

$= \mathcal{L}((6,0), (5,0), (3,-1), (0,-2)) = \mathbb{R}^2$ $\quad \mathcal{L}((1,0),(0,\ldots \ \|$

$\Rightarrow \boxed{\rho(L) = 2,\ \operatorname{Im} L = \mathbb{R}^2}$

$\operatorname{Ker} L = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \right\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} a+2b+3c=0 \\ b+c-2d=0 \end{matrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} b = 2d-c \\ a = -4d-c \end{matrix} \right\} = \left\{ \begin{bmatrix} -4d-c & 2d-c \\ c & d \end{bmatrix} \right\} = \boxed{\mathcal{L}\left\{ \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix} \right\}}$

$\boxed{\delta(L) = 2}$ $\quad + \ \rho(L) + \delta(L) = 4 = \dim M_2(\mathbb{R}) \ +$

$\boxed{4}$ а) $\varphi_A(\lambda) = \det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) \Rightarrow$ с.вредн. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$

с.вектори $\lambda = 1$

$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\begin{matrix} 2b+2c=0 \\ c=0 \end{matrix} \Rightarrow b = 0$

$v = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\lambda = 2$

$\begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow \begin{matrix} -a+2b+2c=0 \\ -b+2c=0 \end{matrix}$

$\Rightarrow b = 2c \quad a = 6c$

$\Rightarrow v = \begin{pmatrix} 6c \\ 2c \\ c \end{pmatrix} \Rightarrow \boxed{v_2 = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}}$

$\Rightarrow \boxed{V = \mathcal{L}((1,0,0), (6,2,1))}$

$V^\perp = \left\{ w_1 = (a,b,c) \middle| \begin{matrix} w \circ v_1 = 0 \\ w \circ v_2 = 0 \end{matrix} \right\} = \left\{ (a,b,c) \middle| \begin{matrix} a = 0 \\ 6a+2b+c=0 \end{matrix} \right\}$

$= \left\{ (a,b,c) \middle| \begin{matrix} a=0 \\ c=-2b \end{matrix} \right\} = \boxed{\mathcal{L}((0,1,-2))}$, $w_1 = (0,1,-2)$

б) $w = f_1 + f_2$, $f_1 \in V$, $f_2 \in V^\perp$

$(1,0,-4) = f_1 + \alpha(0,1,-2) \quad / \circ w_1 \Rightarrow 8 = 5\alpha \Rightarrow \boxed{\alpha = 8/5}$

$\Rightarrow f_2 = \frac{8}{5} w_1 = \left(0, \frac{8}{5}, -\frac{16}{5}\right) \Rightarrow f_1 = w - f_2 = \left(1, -\frac{8}{5}, -\frac{4}{5}\right)$

$d(w, V) = d(w, f_1) = \|f_2\| = \sqrt{0 + \frac{64}{25} + \frac{256}{25}} = \sqrt{\frac{320}{25}}$

$d(w, V^\perp) = d(w, f_2) = \|f_1\| = \sqrt{\frac{25}{25} + \frac{64}{25} + \frac{16}{25}} = \sqrt{\frac{105}{25}}$

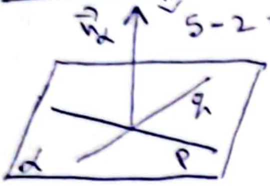
$\Rightarrow w$ је ближи $V^\perp$

[5] $p: \begin{cases} x = -t+3 \\ y = 2t-2 \\ z = t+1 \end{cases}, t \in \mathbb{R}$ $\quad q: \begin{cases} x = s \\ y = 3s-1 \\ z = 5s-2 \end{cases}, s \in \mathbb{R}$

$s = 3-t$
$3s-1 = 2t-2 \Rightarrow 9-3t-1 = 2t-2$
$5s-2 = t+1$

$5t = 10 \Rightarrow \underline{t=2} \Rightarrow s = 1 \Rightarrow$ p и q се секу $\boxed{p \cap q = A(1,2,3)}$

$5-2 = 2+1$ ✓



$p \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{p} = (-1,2,1)$

$q \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{q} = (1,3,5)$

$\Rightarrow \vec{n}_\alpha = \vec{p} \times \vec{q} = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 1 & 3 & 5 \end{vmatrix} = (7,6,-5)$, $A \in p \cap q \subset \alpha$

$\Rightarrow \alpha: 7(x-1) + 6(y-2) - 5(z-3) = 0$

$\boxed{\alpha: 7x + 6y - 5z - 4 = 0}$

[6] $\dim U = \dim W = 4$, $V = M_{2\times3}(\mathbb{R}) \Rightarrow \dim V = 6$,

$V = M_{2\times3}(\mathbb{R}) = \mathcal{L}\left(E_1 = \begin{bmatrix} 1&0&0 \\ 0&0&0 \end{bmatrix}, E_2 = \begin{bmatrix} 0&1&0 \\ 0&0&0 \end{bmatrix}, E_3 = \begin{bmatrix} 0&0&1 \\ 0&0&0 \end{bmatrix}, E_4 = \begin{bmatrix} 0&0&0 \\ 1&0&0 \end{bmatrix}, E_5 = \begin{bmatrix} 0&0&0 \\ 0&1&0 \end{bmatrix}, E_6 = \begin{bmatrix} 0&0&0 \\ 0&0&1 \end{bmatrix}\right)$

1) $\underbrace{\dim(U+W)}_{\leq 6} = \underset{4}{\dim U} + \underset{4}{\dim W} - \dim(U \cap W)$

јер $U+W \leq V$ $\quad \Rightarrow \underline{\dim(U\cap W)} = 8 - \underbrace{\dim(U+W)}_{\leq 6} \underline{\geq 2}$

2) $\left.\begin{matrix} U \cap W \leq U, W \\ \dim U = \dim W = 4 \end{matrix}\right\} \Rightarrow \underline{\dim(U \cap W) \leq 4}$

3) Ако претпоставимо $\dim(U \cap W) = 4$ $\quad$ немамо услов $U \neq W$ па може и ово

$\left.\begin{matrix} U \cap W \leq U, \; U \cap W \leq W \\ \dim U = \dim W = 4 \end{matrix}\right. \Rightarrow \begin{matrix} U \cap W = U \\ U \cap W = W \end{matrix} \Rightarrow U = W$

$\Rightarrow \boxed{\dim U \cap W \in \{2,3,4\}, \dim U+W \in \{4,5,6\}}$

Пример 1) $U = \mathcal{L}(E_1, E_2, E_3, E_4)$
$W = \mathcal{L}(E_1, E_2, E_3, E_5)$
$U \cap W = \mathcal{L}(E_1, E_2, E_3)$
$U + W = \mathcal{L}(E_1, E_2, E_3, E_4, E_5)$
$\dim U \cap W = 3$
$\dim U + W = 5$

2) $U = \mathcal{L}(E_1, E_2, E_3, E_4)$
$W = \mathcal{L}(E_1, E_2, E_5, E_6)$
$U \cap W = \mathcal{L}(E_1, E_2)$
$U + W = M_{2\times3}(\mathbb{R})$
$\dim U \cap W = 2$
$\dim U + W = 6$

3) $U = \mathcal{L}(E_1, E_2, E_3, E_4) = W$
$\Rightarrow U = W = U \cap W = U + W$, $\dim U \cap W = \dim U + W = 4$