ЛААГ - ЈУН 1 - 2023

1 а) База в.п. V је линеарно независан генеришући скуп. Димензија в.п. V је број елемената базе V.

б) Ако је $\vec{v} = \alpha_1\vec{v}_1 + \dots + \alpha_n\vec{v}_n \Rightarrow$ коорд $\vec{v}$ су $[\vec{v}] = (\alpha_1, \dots, \alpha_n)$

в) $L: U \to W \Rightarrow \ker L = \{u \in U \mid L(u) = 0\}$, $\delta(L) = \dim \ker L$

$\operatorname{Im} L = \{w \in W \mid (\exists u \in U)\, L(u) = w\}$, $\rho(L) = \dim \operatorname{Im} L$.

г) Вектор $\vec{v} \neq 0$ је сопств. вектор матрице A ако постоји $\lambda \in \mathbb{R}$ т.д. $A\vec{v} = \lambda\vec{v}$. Скалар λ се тада зове сопствена вредност.

д) Пресл. $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ је скаларни производ ако важи:

1) $\langle \alpha\vec{u} + \beta\vec{v}, \vec{w} \rangle = \alpha\langle \vec{u}, \vec{w} \rangle + \beta\langle \vec{v}, \vec{w} \rangle$

2) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

3) $\langle \vec{u}, \vec{u} \rangle \geq 0$, $\langle \vec{u}, \vec{u} \rangle = 0 \Leftrightarrow \vec{u} = 0$

ђ) (Т) (Кејли-Хамилтон) Матрица A поништава свој карактеристични полином тј. $\varphi_A(A) = 0$.

е) $\vec{a} = (1,2,3)$, $\vec{b} = (-1,0,1) \Rightarrow \cos\angle(\vec{a}, \vec{b}) = \frac{\vec{a} \circ \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{2}{\sqrt{14}\sqrt{2}} = \frac{1}{\sqrt{7}}$

$\Rightarrow \boxed{\angle(\vec{a}, \vec{b}) = \arccos\frac{\sqrt{7}}{7}}$

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{vmatrix} = \boxed{(2, -4, 2)}$

ж) $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_n\vec{v}_n = 0$, A линеарно $\Rightarrow$

$\Rightarrow 0 = A(0) = A(\alpha_1\vec{v}_1 + \dots + \alpha_n\vec{v}_n) = \alpha_1 A\vec{v}_1 + \dots + \alpha_n A\vec{v}_n$

$A\vec{v}_1, \dots, A\vec{v}_n$ линеарно нез $\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0 \Rightarrow \vec{v}_1, \dots, \vec{v}_n$ лин. нез.

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2 У односу на канонску базу $M_2(\mathbb{R})$ имамо

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} U = \mathcal{L}(u_1, u_2, u_3) \\ \dim U = 3 \end{array}}$$

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -9 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} V = \mathcal{L}(v_1, v_2, v_3) \\ \dim V = 3 \end{array}}$$

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & -3 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & -2 & 0 & 1 \\ 0 & 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \boxed{\begin{array}{l} U + W = \mathcal{L}(u_1, u_2, u_3, v_2) \\ \dim(U + W) = 4 \end{array}}$

За пресек посматрамо

$v_1 - u_1 \Rightarrow v_1 - u_1 - u_2 + 2u_1 \Rightarrow v_1 - u_2 + u_1 - u_3 + u_1 = 0$

$v_3 - u_1 \Rightarrow v_3 - u_1 + u_2 - 2u_1 \Rightarrow v_3 - 3u_1 + u_2 + 2u_3 - 2u_1 = 0$

Паметније је било да се одмах пишу врсте али може и на крају.

$\Rightarrow \ v_1 = -2u_1 + u_2 + u_3$
$v_3 = 5u_1 - u_2 - 2u_3$ $\Rightarrow$ $\boxed{\begin{array}{l} U \cap V = \mathcal{L}(v_1, v_3) \\ \dim(U \cap V) = 2 \end{array}}$

Провера $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$
$4 = 3 + 3 - 2 \ \checkmark$

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3 $\varphi_A(\lambda) = \begin{vmatrix} -2-\lambda & -3 & -3 \\ -5 & 4-\lambda & -5 \\ 11 & 3 & 12-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & -3 & 0 \\ -5 & 4-\lambda & \lambda-9 \\ 11 & 3 & 9-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & -3 & 0 \\ 6 & 7-\lambda & 0 \\ 11 & 3 & 9-\lambda \end{vmatrix} = (9-\lambda)\begin{vmatrix} -2-\lambda & -3 \\ 6 & 7-\lambda \end{vmatrix}$

$= (9-\lambda)[\lambda^2 - 5\lambda + 4] = (9-\lambda)(\lambda-4)(\lambda-1) = 0$

$\lambda_1 = 1$ $\quad \lambda_4 = 4$ $\quad \lambda_9 = 9$
$\Rightarrow \vec{v}_1 = (-1,0,1), \ \vec{v}_4 = (-1,1,1), \ \vec{v}_9 = (0,-1,1)$ $\Rightarrow$ $P = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$\Rightarrow \boxed{P^{-1} = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}} \Rightarrow A^n = PD^nP^{-1} = \dots = \boxed{\begin{bmatrix} 2-4^n & 1-4^n & 1-4^n \\ 4^n-9^n & 4^n & 4^n-9^n \\ 9^n+4^n-2 & 4^n-1 & 4^n+9^n-1 \end{bmatrix}}$ ■

4
$2x_1 + \boxed{x_2} + x_3 + 3x_4 = 0 \ / \cdot (-2)$
$3x_1 + 2x_2 + 2x_3 + x_4 = 0$
$x_1 + 2x_2 + 2x_3 - 9x_4 = 0$

$2x_1 + x_2 + x_3 + 3x_4 = 0$
$-x_1 \qquad - 5x_4 = 0$
$-3x_1 \qquad - 15x_4 = 0$

$x_4 = a, a \in \mathbb{R}$
$\Rightarrow x_1 = -5a$
$\Rightarrow x_2 + x_3 = 7a$
$x_3 = b, b \in \mathbb{R}$
$\Rightarrow x_2 = 7a - b$

$\Rightarrow U = \{(-5a, 7a-b, b, a) \mid a, b \in \mathbb{R}\}$
$= \boxed{\mathcal{L}(\underbrace{(-5,7,0,1)}_{e_1}, \underbrace{(0,-1,1,0)}_{e_2})}$

$\Rightarrow \vec{v} = \vec{v}_1 + \vec{v}_2, \ \vec{v}_1 \in U, \ \vec{v}_2 \in U^\perp, \quad \vec{v} = (7,-4,-1,2), \quad \vec{e}_1 = (-5,7,0,1), \ \vec{e}_2 = (0,-1,1,0)$

$\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2 + \vec{v}_2 \quad / \vec{e}_1 \ / \vec{e}_2$
$\vec{v} \circ \vec{e}_1 = \alpha \vec{e}_1 \circ \vec{e}_1 + \beta \vec{e}_2 \circ \vec{e}_1$
$\vec{v} \circ \vec{e}_2 = \alpha \vec{e}_1 \circ \vec{e}_2 + \beta \vec{e}_2 \circ \vec{e}_2$
$\Rightarrow -61 = 75\alpha - 7\beta \ / \cdot 2$
$3 = -7\alpha + 2\beta \ / \cdot 7$
$-101 = 101\alpha \Rightarrow \boxed{\alpha = -1}, \ \boxed{\beta = -2}$

$\Rightarrow \vec{v}_1 = -\vec{e}_1 - 2\vec{e}_2 = \boxed{(5,-5,-2,-1)} \Rightarrow \vec{v}_2 = \vec{v} - \vec{v}_1 = \boxed{(2,1,1,3)}$

ортог. проj. $\vec{v}$ на U

$d(\vec{v}, U) = d(\vec{v}, \vec{v}_1) = \|\vec{v}_2\| = \sqrt{4+1+1+9} = \boxed{\sqrt{15}}$ ■

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5 $p: \begin{array}{l} x = at + 2 \\ y = -3t - 1, \ t \in \mathbb{R} \\ z = 4t + 3 \end{array}, \quad q: \begin{array}{l} x = bs + 3 \\ y = s \\ z = -s + 2 \end{array}$

$\boxed{p \cap q = A(2,-1,3)}$

1) $p \cap q \Rightarrow s = -3t - 1 \Rightarrow 4t + 3 = 3t + 1 + 2 \Rightarrow \boxed{t = 0 \Rightarrow s = -1}$
$\Rightarrow 2 = -b + 3 \Rightarrow \boxed{b = 1}$

2) $p \perp q \Rightarrow 0 = \vec{p} \circ \vec{q} = (a,-3,4) \circ (1,1,-1) = a - 3 - 4 \Rightarrow \boxed{a = 7}$

$\alpha \supset p, q \Rightarrow \vec{n}_\alpha = \vec{p} \times \vec{q} = \begin{vmatrix} i & j & k \\ 7 & -3 & 4 \\ 1 & 1 & -1 \end{vmatrix} = \boxed{(-1, 11, 10)} \Rightarrow \alpha: -(x-2) + 11(y+1) + 10(z-3) = 0$

нпр $\boxed{A(2,-1,3) \in \alpha}$

$\Rightarrow \boxed{\alpha: -x + 11y + 10z - 17 = 0}$ ■

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6 Вежбе, задатак 5.8 ■