ЛААГ - ЈАН 1 - 2023

1 а) $e_1, e_2, \dots, e_k$ лин. независни $\Leftrightarrow \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_k e_k = 0 \Rightarrow \alpha_1 = \dots = \alpha_k = 0$

б) База је линеарно независан генеришући скуп простора V. Димензија је број елемената базе.

в) $L: U \to W$ је линеарно $\Leftrightarrow L(\alpha u + \beta w) = \alpha L(u) + \beta L(w)$, $\forall \alpha, \beta \in \mathbb{R}$, $\forall u, w \in U$

г) Траг матрице ($\operatorname{tr} A$) је збир елемената главне дијаг. матрице A. Матрица B (ако постоји) је инверз матрице A ако важи $AB = BA = E$. Пишемо $B = A^{-1}$.

д) Грасман: $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

ђ) $A \sim B \Leftrightarrow (\exists C)\ A = C^{-1} B C$

$\det A = \det(C^{-1} B C) = \det(C^{-1}) \det B \cdot \det C = \frac{1}{\cancel{\det C}} \det B \cancel{\det C} = \det B$

е) $\vec{AB} = (-1, 1, -2)$ $\quad \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & -2 \\ 2 & 3 & -1 \end{vmatrix} = (5, -5, -5)$

$\vec{AC} = (2, 3, -1)$ $\quad P_{\triangle ABC} = \frac{1}{2} \|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{25+25+25} = \boxed{\frac{5\sqrt{3}}{2}}$

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2 $L: \mathbb{R}^3 \to \mathbb{R}^3[x]$ лин, $L(\underbrace{1,0,1}_{e_1}) = \underbrace{1+2x-x^3}_{f_1}$, $L(\underbrace{1,1,1}_{e_2}) = \underbrace{x+x^2}_{f_2}$, $L(\underbrace{1,2,3}_{e_3}) = f_3 = -1+2x+4x^2+x^3$

а) $\begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ лин. независни $\Rightarrow \exists 1\ L$ које се тражи

$(a,b,c) = x(1,0,1) + y(1,1,1) + z(1,2,3)$

$x + y + z = a$ $\quad x + y + z = a \Rightarrow x = a - a - b + c + \frac{a}{2} - \frac{c}{2} = \boxed{\frac{a}{2} - b + \frac{c}{2}}$

$y + 2z = b$ $\quad y + 2z = b \Rightarrow \boxed{y = a + b - c}$

$x + y + 3z = c$ $\quad 2z = c - a \Rightarrow \boxed{z = \frac{c}{2} - \frac{a}{2}}$

$L(a,b,c) = L\left(\left(\frac{a}{2} - b + \frac{c}{2}\right)e_1 + (a+b-c)e_2 + \left(\frac{c}{2} - \frac{a}{2}\right)e_3\right)$

$\overset{\text{лин}}{=} \left(\frac{a}{2} - b + \frac{c}{2}\right)(1+2x-x^3) + (a+b-c)(x+x^2) + \left(\frac{c}{2} - \frac{a}{2}\right)(-1+2x+4x^2+x^3)$

$= \boxed{a - b + (a-b+c)x + (-a+b+c)x^2 + (-a+b)x^3 = L(a,b,c)}$

б) e_1, e_2, e_3 база $\mathbb{R}^3 \Rightarrow \operatorname{Im} L = \mathcal{L}(L(e_1) = f_1, L(e_2) = f_2, L(e_3) = f_3)$

у станд. бази $(1, x, x^2, x^3)$ од $\mathbb{R}^3[x]$

$\begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array}\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ -1 & 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $\boxed{\rho(L) = 2}$ база $\operatorname{Im} L$ је f_1, f_2

$\operatorname{Ker} L = \{(a,b,c) \in \mathbb{R}^3 \mid L(a,b,c) = 0\} = \{(b,b,0) \mid b \in \mathbb{R}\} = \boxed{\mathcal{L}(1,1,0)}$

$\Rightarrow \boxed{\delta(L) = 1}$

$a - b = 0$
$a - b + c = 0 \Rightarrow c = 0$, $a = b$
$-a + b + c = 0$
$-a + b = 0$

в) $L(e_1) = L(1,0,0) = 1 + x - x^2 - x^3$

$L(e_2) = L(0,1,0) = -1 - x + x^2 + x^3$

$L(e_3) = L(0,0,1) = x + x^2$

$\Rightarrow \boxed{[L]_{E^3}^{E[x]^3} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}}$

$\boxed{3}$

a) $\begin{vmatrix} 1 & 0 & 2 & 3 & 3 \\ -1 & 1 & 2 & -1 & 1 \\ 2 & -1 & \boxed{-1} & 1 & 0 \\ -1 & 3 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 \end{vmatrix} \sim \begin{vmatrix} 5 & -2 & 0 & 5 & 3 \\ 3 & -1 & 0 & 1 & 1 \\ 2 & -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 3 & -2 \\ -2 & -1 & 0 & 1 & 2 \end{vmatrix} = (-1)\cdot(-1)^{3+3}\cdot \begin{vmatrix} 5 & -2 & 5 & 3 \\ 3 & \boxed{-1} & 1 & 1 \\ 1 & 2 & 3 & -2 \\ -2 & -1 & 1 & 2 \end{vmatrix}$

$= -\begin{vmatrix} -1 & -2 & 3 & 1 \\ 0 & -1 & 0 & 0 \\ 7 & 2 & 5 & 0 \\ -5 & -1 & 0 & 1 \end{vmatrix} = -(-1)\cdot(-1)^{2+2}\begin{vmatrix} -1 & 3 & 1 \\ 7 & 5 & 0 \\ -5 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 1 \\ 7 & 5 & 0 \\ -4 & -3 & 0 \end{vmatrix} = -21+20 = \boxed{-1}$

б) $\det A = -1 \neq 0 \Rightarrow \exists A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\,A \Rightarrow \mathrm{adj}\,A = \det A \cdot A^{-1}$

$\det(\mathrm{adj}\,A) = \det(A^{-1}\cdot \det A) \overset{A\in M_5(\mathbb{R})}{=} (\det A)^5 \cdot \det A^{-1} = \frac{(\det A)^5}{\det A} = (\det A)^4 = (-1)^4 = \boxed{1}$

$\boxed{4}$ $V = \mathcal{L}(f_1, f_2, f_3, f_4)$

▲ Проверимо најпре да ли су линеарно независни

$\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array}\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 3 & 7 \\ 1 & 2 & 8 & 6 & 9 \\ 1 & 0 & 4 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 4 & 8 \\ 0 & 2 & 7 & 5 & 8 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 4 & 8 \\ 0 & 0 & -3 & -1 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 4 & 8 \\ 0 & 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{V = \mathcal{L}(f_1, f_2, f_3)}$

$e_1 = f_1 = (1,0,1,1,1)$

$e_2 = f_2 - \frac{f_2 \circ e_1}{e_1 \circ e_1} e_1 = (-1,2,3,3,7) - \frac{12}{4}(1,0,1,1,1) = (-4,2,0,0,4)$

$e_3 = f_3 - \left(\frac{f_3 \circ e_1}{e_1 \circ e_1} e_1 + \frac{f_3 \circ e_2}{e_2 \circ e_2} e_2\right) = (1,2,8,6,9) - \left(\frac{24}{4}(1,0,1,1,1) + \frac{36}{36}(-4,2,0,0,4)\right)$

$= (1,2,8,6,9) - (6,0,6,6,6) - (-4,2,0,0,4) = (-1,0,2,0,-1)$

$\hat{e}_1 = \frac{e_1}{\|e_1\|} = \frac{(1,0,1,1,1)}{\sqrt{4}} = \boxed{\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$

$\hat{e}_2 = \frac{e_2}{\|e_2\|} = \frac{(-4,2,0,0,4)}{\sqrt{16+4+16}} = \boxed{\left(-\frac{2}{3}, \frac{1}{3}, 0, 0, \frac{2}{3}\right)}$

$\hat{e}_3 = \frac{e_3}{\|e_3\|} = \frac{(-1,0,2,0,-1)}{\sqrt{1+4+1}} = \boxed{\left(-\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}\right)}$

$\boxed{\text{ОНБ } V}$

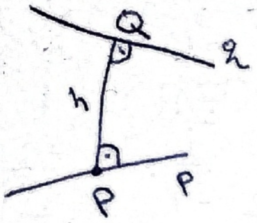
$\boxed{5}$ $p: \frac{x+1}{3} = \frac{y-4}{0} = \frac{z-7}{-1}$, $q: \frac{x+2}{1} = \frac{y+3}{1} = \frac{z+2}{-1}$

▲ Проверимо најпре однос p и q: $\vec{p} = (3,0,-1)$ $\vec{q} = (1,1,-1)$

$P(-1,4,7)$ $Q(-2,-3,-2)$

$[\vec{p}, \vec{q}, \overrightarrow{PQ}] = \left|\begin{matrix} 3 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & -7 & -9 \end{matrix}\right|\begin{matrix} 3 & 0 \\ 1 & 1 \\ -1 & -7 \end{matrix} = -27+7-1-21 \neq 0$

$\Rightarrow$ p и q су мимоилазне



$P(3t-1, 4, -t+7) \in p$ произв. , $\vec{p} = (3,0,-1)$

$Q(s-2, s-3, -s-2) \in q$ произв. , $\vec{q} = (1,1,-1)$

$\overrightarrow{PQ} = (s-3t-1, s-7, -s+t-9)$

1) $\overrightarrow{PQ} \circ \vec{p} = 0 \iff 3s-9t-3+s-t+9=0 \iff \boxed{4s-10t=-6} \cdot 3$

2) $\overrightarrow{PQ} \circ \vec{q} = 0 \iff s-3t-1+s-7+s-t+9=0 \iff \boxed{3s-4t=-1} \cdot -4$ $\downarrow +$

$\Rightarrow -14t = -14 \Rightarrow \boxed{t=1}, \boxed{s=1} \Rightarrow P(2,4,6), Q(-1,-2,-3)$

$\boxed{h = PQ : \frac{x-2}{1} = \frac{y-4}{2} = \frac{z-6}{3}}$ $\Rightarrow \overrightarrow{PQ} = (-3,-6,-9) \parallel \boxed{(1,2,3)}$

$\boxed{6}$ $A \in M_{n\times k}(\mathbb{R})$, $B \in M_{k\times m}(\mathbb{R})$, $\operatorname{rang}(AB) \le \min\{\operatorname{rang} A, \operatorname{rang} B\}$

Ранг матрице је ранг врста (ранг колона)

1)
$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix} \begin{matrix} \leftarrow \vec{b}_1 \\ \leftarrow \vec{b}_2 \\ \\ \leftarrow \vec{b}_k \end{matrix}$$
врсте матрице B

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1k}b_{k1} & a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1k}b_{k2} & a_{11}b_{1m}+\cdots+a_{1k}b_{km} \\ a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2k}b_{k1} & a_{21}b_{12}+a_{22}b_{22}+\cdots+a_{2k}b_{k2} & a_{21}b_{1m}+\cdots+a_{2k}b_{km} \\ \vdots & & \\ a_{n1}b_{11}+a_{n2}b_{21}+\cdots+a_{nk}b_{k1} & a_{n1}b_{12}+a_{n2}b_{22}+\cdots+a_{nk}b_{k2} & a_{n1}b_{1m}+\cdots+a_{nk}b_{km} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11},b_{12},\ldots,b_{1m}) + a_{12}(b_{21},b_{22},\ldots,b_{2m}) + \cdots + a_{1k}(b_{k1},b_{k2},\ldots,b_{km}) \\ a_{21}(b_{11},b_{12},\ldots,b_{1m}) + a_{22}(b_{21},b_{22},\ldots,b_{2m}) + \cdots + a_{2k}(b_{k1},b_{k2},\ldots,b_{km}) \\ \vdots \\ a_{n1}(b_{11},b_{12},\ldots,b_{1m}) + a_{n2}(b_{21},b_{22},\ldots,b_{2m}) + \cdots + a_{nk}(b_{k1},b_{k2},\ldots,b_{km}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\vec{b}_1 + a_{12}\vec{b}_2 + \cdots + a_{1k}\vec{b}_k \\ a_{21}\vec{b}_1 + a_{22}\vec{b}_2 + \cdots + a_{2k}\vec{b}_k \\ \vdots \\ a_{n1}\vec{b}_1 + a_{n2}\vec{b}_2 + \cdots + a_{nk}\vec{b}_k \end{bmatrix}$$

$\Rightarrow$ врсте $AB \in \mathcal{L}(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k)$

$\Rightarrow$ линеарно нез. врста AB има највише колико и лин. нез. врста матрице B (1)

$\Rightarrow \boxed{\operatorname{rang}(AB) \le \operatorname{rang} B}$

2) $\operatorname{rang}(C^T) = \operatorname{rang} C$

$\operatorname{rang}(AB) = \operatorname{rang}(AB)^T = \operatorname{rang}(B^T A^T) \overset{(1)}{\le} \operatorname{rang}(A^T) = \operatorname{rang} A$

$\Rightarrow \boxed{\operatorname{rang}(AB) \le \min\{\operatorname{rang} A, \operatorname{rang} B\}}$

II начин за 2) је да се покаже да колоне AB припадају линеалу колона A