ЛААГ - ЈАН2 - 2024

1 а) $(\exists \alpha,\beta,\gamma \in \mathbb{R})\ \vec{a} = \alpha\vec{b} + \beta\vec{c} + \gamma\vec{d}$

б) adj A је транспонована матрица кофактора

кофактор $A_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} - минор - детерминанта подматрице коју добијемо избацивањем i-врсте и j-те колоне A

в) $L: U \to V \Rightarrow \rho(L) + \delta(L) = \dim U$

г) Пресликавање $\langle , \rangle : V \times V \to V$ је скаларни произв. ако

1) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle$ линеарност по I арг

2) $\langle u,v\rangle = \langle v,u\rangle$ комутат. 3) $\langle u,u\rangle \geq 0$, $\langle u,u\rangle = 0 \Leftrightarrow u=0$

д) $V = U + W \Leftrightarrow (\forall v \in W)(\exists u \in U)(\exists w \in W)\ v = u + w$

ђ) $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

е) $A \sim B \Leftrightarrow (\exists P)\ B = P^{-1}AP$

$$\varphi_B(\lambda) = \det(B-\lambda E) = \det(P^{-1}AP - P^{-1}\lambda E P) = \det(P^{-1}(A-\lambda E)P)$$

$$= \cancel{\det(P^{-1})}\det(A-\lambda E)\cancel{\det P} = \det(A-\lambda E) = \varphi_A(\lambda)$$

ж) $p: x = 1 \Rightarrow \vec{n}_p = (1,0) \Rightarrow \vec{p} = (0,1)$

$q: x - y + 2 = 0 \Rightarrow \vec{n}_q = (1,-1) \Rightarrow \vec{q} = (1,1)$

$$\cos\angle(p,q) = \left|\frac{\vec{p}\circ\vec{q}}{\|\vec{p}\|\cdot\|\vec{q}\|}\right| = \frac{1}{1\cdot\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\angle(p,q) = \frac{\pi}{4}}$$

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2 а) 1) $L(p+q) = \begin{bmatrix}(p+q)(0) & (p+q)(-1)\\ (p+q)(1) & (p+q)(2)\end{bmatrix} = \begin{bmatrix}p(0)+q(0) & p(-1)+q(-1)\\ p(1)+q(1) & p(2)+q(2)\end{bmatrix} \overset{\text{зош на збир 2 матрице}}{=} L(p)+L(q)$

2) $L(\alpha p) = \begin{bmatrix}(\alpha p)(0) & (\alpha p)(-1)\\ (\alpha p)(1) & (\alpha p)(2)\end{bmatrix} = \alpha\begin{bmatrix}p(0) & p(-1)\\ p(1) & p(2)\end{bmatrix} = \alpha L(p)$

$\Rightarrow$ L је линеарно

б) $\mathbb{R}^2[x] = \mathcal{L}(1,x,x^2) \Rightarrow \operatorname{Im}(L) = \mathcal{L}(L(1), L(x), L(x^2))$

$L(1) = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, $L(x) = \begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}$ $L(x^2) = \begin{bmatrix}0 & 1\\ 1 & 4\end{bmatrix}$

коорд у станд. бази $M_2(\mathbb{R}) \Rightarrow$

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & -1 & 1 & 2\\ 0 & 1 & 1 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & -1 & 1 & 2\\ 0 & 0 & 2 & 6\end{bmatrix} \Rightarrow \operatorname{Im} L = \mathcal{L}\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 4\end{bmatrix}\right)$$

$\dim \operatorname{Im} L = \rho(L) = 3$

$\operatorname{Ker} L = \{p = a + bx + cx^2 \mid L(p) = 0\}$

$p(0): a = 0$

$p(-1): a - b + c = 0$

$p(1): a + b + c = 0$

$p(2): a + 2b + 4c = 0$

$\overset{0}{2a} + 2c = 0 \Rightarrow c = 0 \Rightarrow b = 0$

$\Rightarrow \operatorname{Ker}(L) = \{0\}$

$\dim \operatorname{Ker}(L) = \delta(L) = 0$

$\rho(L) + \delta(L) = 3 = \dim \mathbb{R}^2[x]$

3 $\triangle$

$$\varphi_A(\lambda)=\begin{vmatrix}2-\lambda & 1 & -1 & -2\\ 1 & 3-\lambda & -1 & -3\\ -1 & 0 & 2-\lambda & 1\\ 1 & 1 & -1 & -1-\lambda\end{vmatrix} = \begin{vmatrix}2-\lambda & 1 & -1 & -2\\ 1 & 3-\lambda & -1 & -3\\ -1 & 0 & 2-\lambda & 1\\ \lambda-1 & 0 & 0 & 1-\lambda\end{vmatrix} = \begin{vmatrix}2-\lambda & 1 & -1 & -\lambda\\ 1 & 3-\lambda & -1 & -2\\ -1 & 0 & 2-\lambda & 0\\ \boxed{\lambda-1 \quad 0 \quad 0 \quad 0}\end{vmatrix}$$

$$=(\lambda-1)(-1)^{4+1}\begin{vmatrix}1 & -1 & -\lambda\\ 3-\lambda & -1 & -2\\ \boxed{0 \quad 2-\lambda \quad 0}\end{vmatrix} = (1-\lambda)(2-\lambda)(-1)^{3+2}\begin{vmatrix}1 & -\lambda\\ 3-\lambda & -2\end{vmatrix}$$

$$=-(1-\lambda)(2-\lambda)\underbrace{\left(-2+3\lambda-\lambda^2\right)}_{-(\lambda-1)(\lambda-2)} = \boxed{(\lambda-1)^2(\lambda-2)^2}$$

$\Rightarrow \lambda_1=\lambda_2=1$

$(A-E)v=0$

$$\begin{bmatrix}1 & 1 & -1 & -2\\ 1 & 2 & -1 & -3\\ -1 & 0 & 1 & 1\\ 1 & 1 & -1 & -2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=0$$

$$\begin{aligned}a+b-c-2d&=0 \quad /\cdot(-1)\\ a+2b-c-3d&=0\\ -a\quad +c+d&=0\\ a+b-c-2d&=0\end{aligned}$$

$$\begin{aligned}a+b-c-2d&=0\\ b\quad -d&=0 \quad \cdot(-1)\\ b\quad -d&=0\\ 0&=0\end{aligned}$$

$$\begin{aligned}a+b-c-2d&=0\\ b\quad -d&=0\end{aligned}$$

$\Rightarrow d=b$

$\Rightarrow c=a-b$

$$\Rightarrow v=\begin{pmatrix}a\\ b\\ a-b\\ b\end{pmatrix}=\mathcal{L}\left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}\right)$$

$\lambda_3=\lambda_4=2$

$(A-2E)v=0$

$$\begin{bmatrix}0 & 1 & -1 & -2\\ 1 & 1 & -1 & -3\\ -1 & 0 & 0 & 1\\ 1 & 1 & -1 & -3\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=0$$

$$\begin{aligned}b-c-2d&=0\\ a+b-c-3d&=0\\ -a\quad +d&=0 \quad \cdot(-1)\\ a+b-c-3d&=0\end{aligned}$$

$$\begin{aligned}a+b-c-3d&=0\\ b-c-2d&=0\\ 0&=0\\ 0&=0\end{aligned}$$

$\Rightarrow b=c+2d$

$a=d$

$$\Rightarrow v=\begin{pmatrix}d\\ c+2d\\ c\\ d\end{pmatrix}=\mathcal{L}\left(\begin{pmatrix}1\\2\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right)$$

$$P=\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 2 & 1\\ 1 & -1 & 0 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}$$

$$D=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}$$

$\Rightarrow$ 4 сопствена вектора

$\Rightarrow$ јесте дијагоналног типа $\blacksquare$

4 $W=\{(x,y,z,t)\mid x+y=0,\ 2x-z+t=0,\ y+z=0\}$

$=\left(\mathcal{L}((1,1,0,0),(2,0,-1,1),(0,1,1,0))\right)^{\perp}$

а) $\Rightarrow W^{\perp}=\mathcal{L}((1,1,0,0),(2,0,-1,1),(0,1,1,0)) \Rightarrow \dim W^{\perp}=3$

$$\begin{aligned}x+y\quad &=0 \Rightarrow x=-y\\ 2x\quad -z+t&=0 \Rightarrow t=y\\ y+z\quad &=0 \Rightarrow z=-y\end{aligned}$$

$\Rightarrow W=\{(-y,y,-y,y)\mid y\in\mathbb{R}\}=\mathcal{L}(\underbrace{(1,-1,1,-1)}_{w_1})$

$\Rightarrow \dim W=1$

б) $v=(0,0,0,4)=v_1+v_2,\ v_1\in W,\ v_2\in W^{\perp} \Rightarrow v=\alpha w_1+v_2 \quad /\circ w_1$

$\Rightarrow -4=4\alpha \Rightarrow \alpha=-1 \Rightarrow$ $v_1=-w_1=(-1,1,-1,1)$

$v_2=v-v_1=(1,-1,1,3)$

(б) $\cos\angle(v,w)=\cos\angle(v,v_1)=\dfrac{v\circ v_1}{\|v\|\cdot\|v_1\|}=\dfrac{4}{4\cdot\sqrt{4}}=\dfrac{1}{2}$

$\Rightarrow \boxed{\angle(v,w)=\dfrac{\pi}{3}} \Rightarrow$ можемо закључити $\angle(v,w^\perp)=\dfrac{\pi}{2}-\dfrac{\pi}{3}=\dfrac{\pi}{6}$

али ајде да проверимо

$\cos\angle(v,w^\perp)=\cos\angle(v,v_2)=\dfrac{v\circ v_2}{\|v\|\cdot\|v_2\|}=\dfrac{12}{4\cdot\sqrt{12}}=\dfrac{\sqrt{12}}{4}=\dfrac{2\sqrt{3}}{4}=\dfrac{\sqrt{3}}{2}$

$\Rightarrow \boxed{\angle(v,w^\perp)=\dfrac{\pi}{6}} \Rightarrow$ мањи угао је са $w^\perp$ ■

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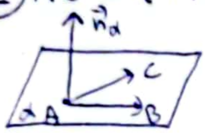
5 $A(-1,0,0)$, $B(0,2,0)$, $C(0,0,1)$, $\alpha\ni A,B,C$, ортоцентар ΔABC.

I) $\vec{AB}=(1,2,0)$, $\vec{AC}=(1,0,1)$

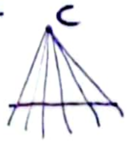


$\Rightarrow \vec n_\alpha=\vec{AB}\times\vec{AC}=\begin{vmatrix} i & j & k\\ 1 & 2 & 0\\ 1 & 0 & 1\end{vmatrix}=(2,-1,-2)$

$A\in\alpha \Rightarrow \alpha: 2(x+1)-y-2z=0$, $\boxed{\alpha: 2x-y-2z+2=0}$

II) $c=AB:\ \dfrac{x+1}{1}=\dfrac{y}{2}=\dfrac{z}{0}=t$ $\qquad AC=b:\ \dfrac{x+1}{1}=\dfrac{y}{0}=\dfrac{z}{1}=s$

Први начин како тражимо h_c:



$AB\ni C'(t-1,2t,0)$ произв.

тражимо t тако да $h_c\perp AB$

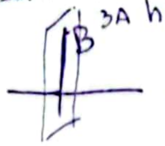
$\Rightarrow \vec{h_c}=\vec{CC'}=(t-1,2t,-1)\perp\vec{AB}=(1,2,0)$

$\Rightarrow t-1+4t=0 \Rightarrow t=\dfrac{1}{5}$

$\Rightarrow \vec{CC'}=\left(-\dfrac{4}{5},\dfrac{2}{5},-1\right)\parallel(-4,2,-5)$

$\Rightarrow \boxed{h_c:\ \dfrac{x}{-4}=\dfrac{y}{2}=\dfrac{z-1}{-5}}=t$

Други начин, тражимо пројекц. B на AC за h_b



β т.д. $B\in\beta$, $\beta\perp AC$

$\Rightarrow \beta: 1(x-0)+0(y-2)+1(z-0)=0$

$\beta: x+z=0$

$B'=\beta\cap AC:\ s-1+s=0$

$\Rightarrow s=\dfrac{1}{2} \Rightarrow B'\left(-\dfrac{1}{2},0,\dfrac{1}{2}\right)$

$\Rightarrow \vec{BB'}=\left(-\dfrac{1}{2},-2,\dfrac{1}{2}\right)\parallel(1,4,-1)$

$\Rightarrow \boxed{h_b:\ \dfrac{x}{1}=\dfrac{y-2}{4}=\dfrac{z}{-1}}=s$

$O=h_c\cap h_b:\ \begin{aligned}-4t&=s &&\Rightarrow s=-4t\\ 2t&=4s+2 &&\to ✓\\ -5t+1&=-s &&\to -5t+1=4t \Rightarrow t=1/9\end{aligned}$ $\Rightarrow s=-4/9$ $\Rightarrow \boxed{O\left(-\dfrac{4}{9},\dfrac{2}{9},\dfrac{4}{9}\right)}$ ■

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6 (а) $x(u+2v+3w)+y(2u+3v+8w)+z(u+2v+4w)=0$

$(x+2y+z)u+(2x+3y+2z)v+(3x+8y+4z)w=0$

u,v,w лин нез $\Rightarrow \begin{aligned}x+2y+z&=0 \quad \cdot(-2),\cdot(-3)\\ 2x+3y+2z&=0\\ 3x+8y+4z&=0\end{aligned}$ $\quad \begin{aligned}x+2y+z&=0\\ -y&=0\\ 2y+z&=0\end{aligned}$ $\Rightarrow \begin{aligned}x&=0\\ y&=0\\ z&=0\end{aligned}$ $\Rightarrow$ $u+2v+3w$, $2u+3v+8w$, $u+2v+4w$ $\boxed{\text{лин. нез.}}$

(б) $\|u\|=3$, $\|u+v\|=4$, $\|u-v\|=6$

$16=\|u+v\|^2=\langle u+v,u+v\rangle=\|u\|^2+2\langle u,v\rangle+\|v\|^2 \Rightarrow 2\langle u,v\rangle+\|v\|^2=7$

$36=\|u-v\|^2=\langle u-v,u-v\rangle=\|u\|^2-2\langle u,v\rangle+\|v\|^2 \Rightarrow -2\langle u,v\rangle+\|v\|^2=27$

сабирањем $\Rightarrow 2\|v\|^2=34 \Rightarrow \boxed{\|v\|=\sqrt{17}}$

$\Rightarrow 2\langle u,v\rangle=7-\|v\|^2=-10 \Rightarrow \boxed{\langle u,v\rangle=-5}$