ЛААГ - ЈАН 1 - 2024

1 а) $V = U \oplus W$ ако се сваки $v \in V$ може јединствено записати као $v = u + w$, $u \in U$, $w \in W$ $\Leftrightarrow$ $V = U + W$ и $U \cap W = \{0\}$.

б) $\mathcal{L}(S) = \mathcal{L}(v_1, v_2, \dots v_n) = \{\alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha_i \in \mathbb{R}\}$

в) База в.п. V је линеарно нез. генеришући скуп простора V. Димензија простора V је број елемената базе.

г) $L: U \to W$ линеарно $\Leftrightarrow$ $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$, $\forall \alpha, \beta \in \mathbb{R}$, $u, v \in U$.

д) Ранг матрице A је број лин. независних врста / колона A.

ђ) (Т) Бине-Коши $\det(AB) = \det A \cdot \det B$

ж) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \varphi_A(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc$
$= \lambda^2 - \operatorname{tr} A \cdot \lambda + \det A$

е) $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0 \quad / \circ v_1$

$\Rightarrow \alpha_1 = 0$, $/ \circ v_2 \Rightarrow \alpha_2 = 0$ ... $\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k = 0 \Rightarrow$ лин. нез.

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2 $U = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, \begin{bmatrix} 4a+2b & -2a-b \\ 4c+2d & -2c-d \end{bmatrix} - \begin{bmatrix} a-2c & b-2d \\ -a+2c & -b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$$\begin{aligned}
3a + 2b + 2c \quad &= 0 \\
-2a - 2b \quad + 2d &= 0 \\
a \quad + 2c + 2d &= 0 \quad /2 \; /-3 \\
b - 2c - 3d &= 0
\end{aligned}$$

$$\begin{aligned}
a \quad + 2c + 2d &= 0 \\
b - 2c - 3d &= 0 \\
-2b + 4c + 6d &= 0 \\
2b - 4c - 6d &= 0
\end{aligned}$$

$\Rightarrow b = 2c + 3d$
$\quad a = -2c - 2d$

$= \left\{ \begin{bmatrix} -2c-2d & 2c+3d \\ c & d \end{bmatrix} \,\middle|\, c, d \in \mathbb{R} \right\}$

$= \mathcal{L}\left(\underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}}_{u_2} \right)$

$\Rightarrow$ $\boxed{u_1, u_2 \text{ лин. нез} \Rightarrow \dim U = 2}$

$W = \mathcal{L}\left(\overset{w_1}{\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}}, \overset{w_2}{\begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}}, \overset{w_3}{\begin{bmatrix} 2 & 0 \\ -3 & 3 \end{bmatrix}} \right)$

посматрамо коорд. у станд. бази $M_2(\mathbb{R})$

$\Rightarrow \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \begin{bmatrix} 2 & -1 & -2 & 3 \\ -2 & 2 & 1 & -3 \\ 2 & 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \Rightarrow$ $\boxed{W = \mathcal{L}(w_1, w_2), \; \dim W = 2}$

$U + W$: $\begin{array}{c} u_1 \\ u_2 \\ w_1 \\ w_2 \end{array} \begin{bmatrix} -2 & 2 & 1 & 0 \\ -2 & 3 & 0 & 1 \\ 2 & -1 & -2 & 3 \\ -2 & 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ u_2 - u_1 \;/-1 \\ w_1 + u_1 \\ w_2 - u_1 \end{array} \sim \begin{bmatrix} -2 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ \\ \cdot \frac{3}{2} \\ \end{array}$

$w_1 + 2u_1 - u_2$

$\Rightarrow$ $\boxed{\dim(U+W) = 3, \; U + W = \mathcal{L}(u_1, u_2, w_1)}$

$\frac{3}{2}(w_1 + 2u_1 - u_2) + w_2 - u_1 = 0 \Rightarrow \underbrace{2u_1 - \frac{3}{2}u_2}_{\in U} = \underbrace{-\frac{3}{2}w_1 - w_2}_{\in W}$

$\Rightarrow \dim U \cap W = 1$

$U \cap W = \mathcal{L}\left(2u_1 - \frac{3}{2}u_2\right) = \mathcal{L}\left(\begin{bmatrix} -1 & -1/2 \\ 2 & -3/2 \end{bmatrix} \right)$

$\boxed{3}$ $L : \mathbb{R}^2[x] \to \mathbb{R}^2[x]$, $L(p) = p(x+1) + p(x)$

а) $[L]_F$, $F = \{ f_1 = 1,\ f_2 = 1+x,\ f_3 = 1 + x + x^2 \}$ б) $L^{-1} = ?$

а) $L(f_1) = L(1) = 1 + 1 = 2 = 2f_1$

$L(f_2) = L(1+x) = 1 + (x+1) + 1 + x = 3 + 2x = f_1 + 2f_2$

$L(f_3) = L(1+x+x^2) = 1 + (x+1) + \overset{1+2x+x^2}{(x+1)^2} + 1 + x + x^2$

$= 4 + 4x + 2x^2 = 2f_2 + 2f_3$

$\Rightarrow \boxed{[L]_F = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}}$

б) $L(ax^2+bx+c) = a(x+1)^2 + b(x+1) + c + ax^2 + bx + c$

$= \underbrace{2a}_{=p}x^2 + \underbrace{(2a+2b)}_{=q}x + \underbrace{a+b+2c}_{=r}$

$$\begin{aligned} a + b + 2c &= r \\ 2a + 2b &= q \\ 2a &= p \end{aligned}$$

$L(ax^2+bx+c) = px^2 + qx + r$

$\Rightarrow L^{-1}(px^2+qx+r) = ax^2+bx+c$

$\Rightarrow a = p/2$

$b = \frac{1}{2}(q-p)$

$c = \frac{1}{2}(r - \frac{1}{2}q)$

$\Rightarrow \boxed{L^{-1}(px^2+qx+r) = \frac{p}{2}x^2 + (\frac{1}{2}q - \frac{1}{2}p)x + (\frac{1}{2}r - \frac{1}{4}q)}$

$\operatorname{Ker} L = \{ ax^2+bx+c \mid 2a=0,\ 2a+2b=0,\ a+b+2c=0 \} = \{0\}$

$\boxed{4}$ $V = \mathcal{L}(f_1, f_2, f_3, f_4) = \mathcal{L}(f_1, f_2, f_3)$, $f_1 = (1,2,0,3)$, $f_2 = (4,0,5,8)$, $f_3 = (8,1,5,6)$

$e_1 = f_1 = (1,2,0,3)$

$e_2 = f_2 - \frac{f_2 \circ e_1}{e_1 \circ e_1} e_1 = (4,0,5,8) - \frac{28}{14}(1,2,0,3) = (2,-4,5,2)$

$e_3 = f_3 - \left(\frac{f_3 \circ e_1}{e_1 \circ e_1} e_1 + \frac{f_3 \circ e_2}{e_2 \circ e_2} e_2 \right) = (8,1,5,6) - \left(\frac{28}{14}(1,2,0,3) + \frac{49}{49}(2,-4,5,2) \right)$

$= (8,1,5,6) - (4,0,5,8) = (4,1,0,-2)$

$\Rightarrow \boxed{\hat{e}_1 = \frac{e_1}{\|e_1\|} = \frac{(1,2,0,3)}{\sqrt{14}},\ \hat{e}_2 = \frac{(2,-4,5,2)}{7},\ \hat{e}_3 = \frac{(4,1,0,-2)}{\sqrt{21}}}$

$\boxed{5}$ $A(-1,0,-4)$, $B(7,2,2)$, $C(2,1,3)$, $D(0,-1,1)$

$\overrightarrow{AB} = (8,2,6) \parallel (4,1,3)$, $\overrightarrow{CD} = (-2,-2,-2) \parallel (1,1,1)$

$\Rightarrow AB : \frac{x+1}{4} = \frac{y}{1} = \frac{z+4}{3}$, $CD : \frac{x}{1} = \frac{y+1}{1} = \frac{z-1}{1}$

$P(4t-1, t, 3t-4) \in AB$ произв, $Q(s, s-1, s+1) \in CD$ произв

$\Rightarrow \overrightarrow{PQ} = (s-4t+1,\ s-t-1,\ s-3t+5)$

1) $\overrightarrow{PQ} \perp \overrightarrow{AB} \iff 4s-16t+4+s-t-1+3s-9t+15=0 \iff 8s-26t=-18$

2) $\overrightarrow{PQ} \perp \overrightarrow{CD} \iff s-4t+1+s-t-1+s-3t+5=0 \iff \underline{3s-8t=-5}$

$$\begin{aligned} 3s-8t&=-5 \quad /\cdot(-4) \\ 4s-13t&=-9 \quad /\cdot 3 \end{aligned} \qquad -7t=-7 \implies \boxed{t=1}, \ \underline{s=1}$$

$\implies P(3,1,-1),\ Q(1,0,2) \implies \overrightarrow{PQ}=(-2,-1,3)$

1) $n: \dfrac{x-1}{-2}=\dfrac{y}{-1}=\dfrac{z-2}{3}$, 2) $d(AB,CD)=\|\overrightarrow{PQ}\|=\sqrt{14}$

$\boxed{6}$ а) $A: V \to V$ линеарно, $\dim V < +\infty$, $\operatorname{Ker} A + \operatorname{Im} A = V \iff \operatorname{Ker} A \cap \operatorname{Im} A = \{0\}$

б) пример т.д. $\operatorname{Ker} A + \operatorname{Im} A \neq V$

а) $\boxed{\Rightarrow}$ Нека је $\operatorname{Ker} A + \operatorname{Im} A = V$ теорема о рангу и дефекту, $\rho(A)+\delta(A)=\dim V$

$\implies \dim(\operatorname{Ker} A + \operatorname{Im} A) = \dim V \overset{\text{T}}{=} \dim(\operatorname{Ker} A) + \dim(\operatorname{Im} A)$

Грасманова ф-ла $\implies \dim(\operatorname{Ker} A \cap \operatorname{Im} A) = 0 \implies \operatorname{Ker} A \cap \operatorname{Im} A = \{0\}$

$\boxed{\Leftarrow}$ Нека је $\operatorname{Ker} A \cap \operatorname{Im} A = \{0\} \implies \dim(\operatorname{Ker} A \cap \operatorname{Im} A) = 0$

Грасман ф-ла $\implies \dim(\operatorname{Ker} A + \operatorname{Im} A) = \dim(\operatorname{Ker} A) + \dim(\operatorname{Im} A)$

$$= \delta(A) + \rho(A) = \dim V$$

$$\implies \operatorname{Ker} A + \operatorname{Im} A = V$$

б) $A: \mathbb{R}^2 \to \mathbb{R}^2$, $A(x,y) = (y,0)$

$\implies \operatorname{Im} A = \{(x,0) \mid x \in \mathbb{R}\} = \operatorname{Ker} A$

$\implies \operatorname{Im} A + \operatorname{Ker} A = \{(x,0) \mid x \in \mathbb{R}\} \neq \mathbb{R}^2$ ∎
