1. КОЛОКВИЈУМ 2023

1 (а) Вектори $e_1, e_2, \dots, e_n$ су <u>линеарно независни</u> ако

$$\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

(б) <u>Инверз матрице</u> $A \in M_n(\mathbb{R})$ је матрица $B \in M_n(\mathbb{R})$, ако постоји, т.д. важи $AB = BA = E$. Пишемо $B = A^{-1}$.

(в) <u>Сума в.п. U и W</u> је в.п. $U + W = \{u + w \mid u \in U, w \in W\}$

(г) Пресликавање $L : U \to W$ је <u>линеарно</u> ако задовољава
1) $(\forall u_1, u_2 \in U)\ L(u_1 + u_2) = L(u_1) + L(u_2)$; 2) $(\forall \alpha \in \mathbb{R})(\forall u \in U)\ L(\alpha u) = \alpha L(u)$

(д) $\vec{a} = (7, 6, -6)$, $\vec{b} = (6, 2, 9)$

$$\cos\alpha = \frac{\vec{a} \circ \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \frac{7 \cdot 6 + 2 \cdot 6 + 9 \cdot (-6)}{-\|-} = 0$$

$$\Rightarrow \vec{a} \perp \vec{b} \text{ тј } \boxed{\measuredangle(\vec{a}, \vec{b}) = \frac{\pi}{2}}$$

$$d(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\| = \|(1, 4, -15)\|$$

$$= \sqrt{1 + 16 + 225} = \sqrt{242} = \boxed{11\sqrt{2}}$$

$\vec{c} = (-6, 9, 2)$, $\vec{d} = (7, 6, -6)$

$$\vec{c} \circ \vec{d} = (-6) \cdot 7 + 9 \cdot 6 + 2 \cdot (-6) = 0$$

$$\Rightarrow \vec{c} \perp \vec{d} \Rightarrow \boxed{\measuredangle(\vec{c}, \vec{d}) = \pi/2}$$

$$d(\vec{c}, \vec{d}) = \|\vec{c} - \vec{d}\| = \|(-13, 3, 8)\|$$

$$= \sqrt{169 + 9 + 64} = \sqrt{242} = \boxed{11\sqrt{2}}$$

---

2 Решити систем Гаусовом методом (група ГАУС)

$$\begin{aligned}
(a+2)x + y \quad &= a - 1 \\
x - ay + z &= 1 - 2a \quad \leftarrow + \\
ax - y + \boxed{z} &= -1 \quad / \cdot (-1)
\end{aligned}$$

$$\begin{aligned}
ax - y + z &= -1 \\
(a+2)x + \boxed{y} \quad &= a - 1 \quad / \cdot (a-1) \\
(1-a)x + (1-a)y &= 2 - 2a \quad \leftarrow +
\end{aligned}$$

$$\boxed{\begin{aligned}
ax - y + z &= -1 \\
(a+2)x + y \quad &= a - 1
\end{aligned}}$$

$$((a+2)(a-1) + (1-a))x = (a-1)^2 + 2(1-a)$$

$$\boxed{(a-1)(a+1)x = (a-1)(a-3)}$$

I $a \neq 1$, $a \neq -1$

$$x = \frac{(a-1)(a-3)}{(a-1)(a+1)} = \frac{a-3}{a+1}$$

$$y = a - 1 - \frac{(a+2)(a-3)}{a+1} = \frac{a+5}{a+1}$$

$$z = -1 - \frac{(a-3)a}{a+1} + \frac{a+5}{a+1}$$

$$= \frac{-a - 1 - a^2 + 3a + a + 5}{a+1} = \frac{-(a-4)(a+1)}{a+1}$$

$$\boxed{(x, y, z) = \left(\frac{a-3}{a+1}, \frac{a+5}{a+1}, 4 - a\right)}$$

$a \neq 1$, $a \neq -1$
јединствено решење

II $a = -1 \Rightarrow 0 = 8$ ↯ $\Rightarrow$ <u>нема решења за $a = -1$</u>

III $a = 1 \Rightarrow$

$$\begin{aligned}
x - y + z &= -1 \Rightarrow z = -1 - \alpha - 3\alpha = -4\alpha - 1 \\
3x + y \quad &= 0 \Rightarrow x = \alpha, \alpha \in \mathbb{R} \Rightarrow y = -3\alpha \\
0 &= 0
\end{aligned}$$

$$\Rightarrow \boxed{(x, y, z) = (\alpha, -3\alpha, -4\alpha - 1),\ \alpha \in \mathbb{R},\ a = 1}$$

бесконачно решења

\+ Група САРУС: исти систем за $b = a$ и промену x и y променљиве

3 а) $U = \{p \in \mathbb{R}^3[x] \mid p(0) = p(1)\}$

$= \{a + bx + cx^2 + dx^3 \mid a = a + b + c + d\}$

$= \{a + bx + cx^2 + (-b-c)x^3\} = \{a + b(x - x^3) + c(x^2 - x^3) \mid a,b,c \in \mathbb{R}\}$

$\mathcal{L}(\underbrace{1}_{p_1}, \underbrace{x - x^3}_{p_2}, \underbrace{x^2 - x^3}_{p_3}) \Rightarrow$ $\boxed{\dim U = 3}$, база p_1, p_2, p_3

ГРУПА ГАУС
+ ГРУПА САРУС само обрнути U и W +

б) $W = \{p \in \mathbb{R}^3[x] \mid p(0) = p'(0) = p''(0) = 0\}$

$= \{a + bx + cx^2 + dx^3 \mid a = 0,\ b = 0,\ 2c = 0\} = \mathcal{L}(\underbrace{x^3}_{r_1})$,

база је r_1
$\dim W = 1$

$V = U \oplus W$? $\Leftrightarrow$ $\dim U + W = \dim U + \dim W = 4 = \dim \mathbb{R}^3[x]$

Довољно је доказати $\dim U + W = 4$:

Посматрамо коорд. базних вектора у станд. бази $(1, x, x^2, x^3)$ од $\mathbb{R}^3[x]$

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ r_1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

p_1, p_2, p_3, r_1 су линеарно независни

$\Rightarrow$ чине базу $U + W$ $\Rightarrow$ $\dim U + W = 4$

$\Rightarrow$ сума је директна ■

ГАУС

4 $F: M_2(\mathbb{R}) \to \mathbb{R}^3$, $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + 3c + 2d,\ 2a + 4b + 7c + 5d,\ a + 2b + 6c + 5d)$

а) $M_2(\mathbb{R}) = \mathcal{L}\left(E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \Rightarrow \operatorname{Im} F = \mathcal{L}(FE_1, FE_2, FE_3, FE_4)$

$FE_1 = (1,2,1) = f_1$

$FE_2 = (2,4,2) = f_2$

$FE_3 = (3,7,6) = f_3$

$FE_4 = (2,5,5) = f_4$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 6 \\ 2 & 5 & 5 \end{bmatrix} \xrightarrow{-2,\,-3,\,-2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$\operatorname{Im} F = \mathcal{L}(FE_1, FE_3)$

$\rho(F) = 2$

б) $\operatorname{Ker} F = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \mid F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0\right\} = (*)$

$$\begin{aligned} a + 2b + 3c + 2d &= 0 \quad \cdot(-2),\ \cdot(-1) \\ 2a + 4b + 7c + 5d &= 0 \quad + \\ a + 2b + 6c + 5d &= 0 \quad + \end{aligned}$$

$a + 2b + 3c + 2d = 0 \Rightarrow a = -2b - c$

$\left.\begin{aligned} c + d &= 0 \\ 3c + 3d &= 0 \end{aligned}\right\} \Rightarrow d = -c$

$(*) = \left\{\begin{bmatrix} -2b - c & b \\ c & -c \end{bmatrix} \mid b, c \in \mathbb{R}\right\} = \mathcal{L}\left(\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\right), \delta(F) = 2$

САРУС Идентификацијом $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ са њеним коорд. у ст. бази (a,b,c,d) потпуно исто добијамо решење ■

---

5 ГАУС а) $\det A = -a + 9 \neq 0 \Rightarrow \exists A^{-1}$ за $a \neq 9$

б)
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2,\,-4} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \xrightarrow{+} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}}$$

$\boxed{\text{САРУС}}$ а) $\det B = -b+4 \neq 0 \Rightarrow \exists B^{-1}$ за $b \neq 4$

б) $B^{-1} = \frac{1}{\det B} \operatorname{adj} B$, за $b=3 \Rightarrow \det B = 1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \xrightarrow{\text{минори}} \begin{bmatrix} \begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix} \xrightarrow{\text{минори}} \begin{bmatrix} -11 & 4 & 6 \\ -2 & 0 & 1 \\ +2 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{кофакт}} \begin{bmatrix} -11 & -4 & 6 \\ 2 & 0 & -1 \\ +2 & 1 & -1 \end{bmatrix} \xrightarrow{\text{трансп.}} \underbrace{\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}}_{\operatorname{adj} B} \Rightarrow \boxed{B^{-1} = \frac{1}{\det B} \operatorname{adj} B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}}$$

$\boxed{\text{ГАУС}}$

$\boxed{6}$ $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$

а)

$$tr(AB) = (\text{само дијагонала } AB) = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}$$
$$+ a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}$$
$$+ \cdots$$
$$+ a_{m1}b_{1m} + a_{m2}b_{2m} + \cdots + a_{mn}b_{nm}$$

$$tr(BA) = (\text{само дијагонала } BA) = b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1m}a_{m1}$$
$$+ b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2m}a_{m2}$$
$$+ \cdots$$
$$+ b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nm}a_{mn}$$

Прегруписавањем чланова видимо да су „колоне збира у AB" једнаке „врстама збира у BA" $\Rightarrow tr(AB) = tr(BA)$

б) $tr(BAB^{-1}) = tr((BA)B^{-1}) \overset{а}{=} tr(B^{-1}(BA)) = tr(B^{-1}BA) = tr A$

$\boxed{\text{САРУС}}$ Исто за $P = A$ и $Q = B$