$\boxed{\text{ЛААГ 1.КОЛ 2022}}$

$\boxed{1}$ ⓐ $\mathcal{L}\{v_1,\dots,v_k\} = \mathcal{L}(v_1,\dots,v_k) = \{\alpha_1 v_1 + \dots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$

ⓑ V је сума простора U и W, тј. $V = U + W$ ако важи $(\forall v \in V)(\exists u \in U, w \in W)\ v = u + w$. Ако је ово представљање јединствено онда је сума директна, тј. $V = U \oplus W$.

ⓒ $V = U \oplus W \iff V = U + W$ и $U \cap W = \{0\}$.

ⓓ Ранг матрице A је број линеарно независних врста (колона) матрице A (или ред највеће поддетерминанте од A различите од нула).

ⓔ $L: W \to U$, $\operatorname{Ker} L = \{w \in W \mid L(w) = 0\}$, $\dim \operatorname{Ker} L = \delta(L)$

$\operatorname{Im} L = \{u \in U \mid (\exists w \in W)\ L(w) = u\} = L(W)$, $\dim \operatorname{Im} L = \rho(L)$

ⓓ $\operatorname{Ker} L \leq W$? $\quad$ $\underline{L: W \to U \text{ линеарно}}$ $\quad$ $\operatorname{Im} L \leq U$?

1) $0 \in \operatorname{Ker} L$ јер $L(0) = 0$ ✓

2) $w_1, w_2 \in \operatorname{Ker} L \Rightarrow w_1 + w_2 \in \operatorname{Ker} L$?

$L(w_1 + w_2) = \overset{=0}{L(w_1)} + \overset{=0}{L(w_2)} = 0$ ✓

3) $w \in \operatorname{Ker} L, \alpha \in \mathbb{R} \Rightarrow \alpha w \in \operatorname{Ker} L$?

$L(\alpha w) = \alpha \cdot L(w) = \alpha \cdot 0 = 0$ ✓

$\Rightarrow \boxed{\operatorname{Ker} L \leq W}$

1) $0 \in \operatorname{Im} L$ јер $L(0) = 0$ ✓

2) $\overset{L(w_1)}{u_1}, \overset{L(w_2)}{u_2} \in \operatorname{Im} L \Rightarrow u_1 + u_2 \in \operatorname{Im} L$?

$u_1 + u_2 = L(w_1) + L(w_2) = L(w_1 + w_2)$ ✓

3) $\overset{L(w)}{u} \in \operatorname{Im} L, \alpha \in \mathbb{R} \Rightarrow \alpha u \in \operatorname{Im} L$?

$\alpha \cdot u = \alpha \cdot L(w) = L(\alpha w)$ ✓

$\Rightarrow \boxed{\operatorname{Im} L \leq U}$

$\boxed{2}$ $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$. $U = \{X \in M_2(\mathbb{R}) \mid AX + XB = 2X^T\}$

ⓐ $U \leq M_2(\mathbb{R})$?

1) $0 \in U$ јер $0 = A \cdot 0 + 0 \cdot B = 2 \cdot 0^T = 0$ ✓

2) $X, Y \in U \Rightarrow X + Y \in U$?

$A(X+Y) + (X+Y)B = AX + AY + XB + YB \overset{X,Y \in U}{=} 2X^T + 2Y^T = 2(X+Y)^T$ ✓

3) $X \in U, \alpha \in \mathbb{R} \Rightarrow \alpha X \in U$?

$A(\alpha X) + (\alpha X)B = \alpha(AX + XB) = \alpha \cdot 2X^T = 2(\alpha X)^T$ ✓ $\Rightarrow U \leq M_2(\mathbb{R})$

ⓑ $U = \left\{ X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} = 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \right\}$

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{bmatrix} a+2c & b+2d \\ 3c & 3d \end{bmatrix} + \begin{bmatrix} -b & 2a+b \\ -d & 2c+d \end{bmatrix} = \begin{bmatrix} 2a & 2c \\ 2b & 2d \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{bmatrix} a+2c-b & 2b+2d+2a \\ 3c-d & 2c+4d \end{bmatrix} = \begin{bmatrix} 2a & 2c \\ 2b & 2d \end{bmatrix} \right\}$

$$\begin{aligned} a+2c-b &= 2a \\ 2b+2d+2a &= 2c \\ 3c-d &= 2b \\ 2c+4d &= 2d \end{aligned} \Rightarrow \begin{aligned} -a-b+2c &= 0 \Rightarrow a = 0 \\ 2a+2b-2c+2d &= 0 \ ✓ \\ -2b+3c-d &= 0 \Rightarrow b = 2c \\ 2c+2d &= 0 \Rightarrow d = -c \end{aligned}$$

$\Rightarrow \boxed{U = \left\{ \begin{bmatrix} 0 & 2c \\ c & -c \end{bmatrix} \middle| c \in \mathbb{R} \right\} = \mathcal{L}\left(\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\right)}$

$\Rightarrow \boxed{\dim U = 1}$ $\quad$ (део под а) се може одрадити и овако)

B) $W = \{ X \in M_2(\mathbb{R}) \mid X^T = X \} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \,\middle|\, a, b, d \in \mathbb{R} \right\}$

1) $M_2(\mathbb{R}) = U + W$?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 2\square \\ \square & -\square \end{bmatrix} + \begin{bmatrix} * & \# \\ \# & \star \end{bmatrix}$$

I) $\boxed{* = a}$

II) $2\square + \# = b$, $\square + \# = c \;/\cdot(-1)\,+ \Rightarrow \boxed{\square = b - c}$, $\boxed{\# = 2c - b}$

III) $-\square + \star = d \Rightarrow \boxed{\star = b - c + d}$

Након кратке игре, добијамо

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2b-2c \\ b-c & c-b \end{bmatrix}}_{\in U} + \underbrace{\begin{bmatrix} a & 2c-b \\ 2c-b & b-c+d \end{bmatrix}}_{\in W} \Rightarrow \boxed{M_2(\mathbb{R}) = U + W}$$

2) $U \cap W = ?$ $\quad U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a = 0,\ b = 2c,\ d = -c \right\}$, $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, b = c \right\}$

$U \cap W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a = 0,\ b = 2c,\ d = -c,\ b = c \right\}$

$\begin{matrix} b = 2c \\ b = c \end{matrix} \Rightarrow \boxed{c = 0} \Rightarrow \boxed{b = 0} \Rightarrow \boxed{d = 0} \Rightarrow \boxed{U \cap W = \{0\}}$

1) и 2) $\Rightarrow M_2(\mathbb{R}) = U \oplus W$

$\boxed{3}$ $U = \mathcal{L}((1,0,0,1),(1,1,1,2),(-2,0,1,1))$

$W = \mathcal{L}((1,0,-2,3),(3,1,-3,7),(7,3,-5,15))$

$\boxed{U:}$ $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ -2 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow$ $\boxed{U = \mathcal{L}(u_1, u_2, u_3) \text{ лин. нез.},\ \dim U = 3}$

$\boxed{W:}$ $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 3 & 1 & -3 & 7 \\ 7 & 3 & -5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $\boxed{W = \mathcal{L}(w_1, w_2),\ \dim W = 2}$

$\boxed{U+W:}$

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & -2 & 3 \\ 3 & 1 & -3 & 7 \end{bmatrix} \sim \begin{array}{r} u_1 \\ u_2 - u_1 \\ u_3 + 2u_1 \\ w_1 - u_1 \\ w_2 - 3u_1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 1 & -3 & 4 \end{bmatrix} \quad \begin{array}{r} u_1 \\ u_2 - u_1 \\ u_3 + 2u_1 \\ w_1 - u_1 \\ w_2 - 2u_1 - u_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix} \begin{array}{l} \\ \\ \\ w_1 + 3u_1 + 2u_3 \quad / \cdot(-15) \\ w_2 + 6u_1 - u_2 + 4u_3 \quad / \cdot 8 \Rightarrow 0 \end{array} \Rightarrow \boxed{\begin{array}{l} U + W = \mathcal{L}(u_1, u_2, u_3, w_1) \\ \dim U + W = 4 \end{array}}$$

* $\underbrace{\dim(U+W)}_{=4} = \underbrace{\dim U}_{=3} + \underbrace{\dim W}_{=2} - \dim(U \cap W) \Rightarrow U + W = \mathbb{R}^4$

$\Rightarrow \boxed{\dim(U \cap W) = 1}$

Из последњих релација у $U+W$ имамо

$$-15(w_1+3u_1+2u_3)+8(w_2+6u_1-u_2+4u_3)=0$$

$$\underbrace{3u_1-8u_2+2u_3}_{\in U}=\underbrace{15w_1-8w_2}_{\in W}=(-9,-8,-6,-11) \Rightarrow \boxed{U\cap W=\mathcal{L}((9,8,6,11))}$$

$\boxed{4}$ $L:\mathbb{R}^2[x]\to\mathbb{R}^2[x]$, $L(a+bx+cx^2)=(3b-3c)-(a-4b+3c)x-(a-3b+2c)x^2$

а) 1) $\mathbb{R}^2[x]=\mathcal{L}(e_1=1,e_2=x,e_3=x^2) \Rightarrow \operatorname{Im} L=\mathcal{L}(L(1),L(x),L(x^2))$

$L(1)=-x-x^2=(0,-1,-1)_E=l_1$

$L(x)=3+4x+3x^2=(3,4,3)_E=l_2 \Rightarrow \boxed{\operatorname{Im} L=\mathcal{L}(l_2,l_3) \quad \rho(L)=2}$

$L(x^2)=-3-3x-2x^2=(-3,-3,-2)_E=l_3$

$$\begin{array}{c} l_2\\ l_3\\ l_1\end{array}\begin{bmatrix}3&4&3\\-3&-3&-2\\0&-1&-1\end{bmatrix}\sim\begin{bmatrix}3&4&3\\0&1&1\\0&-1&-1\end{bmatrix}\sim\begin{bmatrix}3&4&3\\0&1&1\\0&0&0\end{bmatrix}$$

2) $\operatorname{Ker} L=\{a+bx+cx^2 \mid L(a+bx+cx^2)=0\}=\boxed{\mathcal{L}(1+x+x^2)=\operatorname{Ker} L}$ $\boxed{\delta(L)=1}$

$3b-3c=0$ $\qquad a-3b+2c=0 \Rightarrow \boxed{a=c}$

$a-4b+3c=0$ $\qquad 3b-3c=0$

$\boxed{a}-3b+2c=0 \;\cdot(-1)$ $\qquad -b+c=0 \Rightarrow \boxed{b=c}$

б) I НАЧИН

$$[L]_E=\begin{bmatrix}0&3&-3\\-1&4&-3\\-1&3&-2\end{bmatrix},\quad P_{EF}=\begin{bmatrix}1&3&-3\\1&1&0\\1&0&1\end{bmatrix}\Rightarrow\ldots\Rightarrow P^{-1}=\begin{bmatrix}1&-3&3\\-1&4&-3\\-1&3&-2\end{bmatrix}$$

$$[L]_F=P^{-1}[L]_EP=\begin{bmatrix}1&-3&3\\-1&4&-3\\-1&3&-2\end{bmatrix}\begin{bmatrix}0&3&-3\\0&1&0\\0&0&1\end{bmatrix}=\boxed{\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}=[L]_F}$$

II НАЧИН

$L(f_1)=0_E=0_F=(0,0,0)_F$ $\quad f_1\in\operatorname{Ker} L$

$L(f_2)=L(3+x)=3+x=f_2=(0,1,0)_F$

$L(f_3)=L(-3+x^2)=-3+x^2=(0,0,1)_F$

Овде је много лакши II начин јер се одмах добију у слици вектори из базе F

$\boxed{5}$

$x-y+2z=-3$

$ax+2y-3z=5-a \Rightarrow\ldots$

$2x+ay-z=1$

$$\boxed{\begin{aligned}\Delta&=2(a+2)(a-1)\\ \Delta_x&=-2a(a-1)\\ \Delta_y&=4(a-1)\\ \Delta_z&=-2(a+2)(a-1)\end{aligned}}$$

I) $\boxed{a\neq1,\ a\neq-2}$ $\Rightarrow \Delta\neq0 \Rightarrow$ јед. решење

$$\boxed{x=\frac{\Delta_x}{\Delta}=\frac{-a}{a+2}\quad y=\frac{\Delta_y}{\Delta}=\frac{2}{a+2}\quad z=\frac{\Delta_z}{\Delta}=-1}$$

II) $\boxed{a=-2}$

$\Delta=0$

$\Delta_y=-12\neq0$ $\Rightarrow$ $\boxed{\text{НЕМА РЕШЕЊА}}$

III) $\boxed{a=1}$ $\Delta=0=\Delta_x=\Delta_y=\Delta_z$

$\boxed{x}-y+2z=-3 \;/\cdot(-1)$ $\quad\Leftrightarrow\quad x-y+2z=-3$

$x+2y-3z=4$ $\qquad 3y-5z=7$

$2x+y-z=1$ $\qquad 3y-5z=7$

$\Rightarrow z=\alpha,\ \alpha\in\mathbb{R} \Rightarrow y=\frac{7+5\alpha}{3}$

$x=-3+\frac{7+5\alpha}{3}-2\alpha$

$$\boxed{R_\alpha=\left\{(x,y,z)=\left(\frac{-2-\alpha}{3},\frac{7+5\alpha}{3},\alpha\right) \,\middle|\, \alpha\in\mathbb{R}\right\}}$$

$\boxed{6}$ $\vec{a}=(1,a,a^2)$, $\vec{b}=(1,b,b^2)$, $\vec{c}=(1,c,c^2)$ лин. независни $\iff a\neq b\neq c\neq a$.

$\boxed{\Rightarrow}$ ПС нека је нпр $a=b$ (аналогно за остале једнакости)

$\Rightarrow \vec{a}-\vec{b}=0 \Rightarrow \vec{a},\vec{b}$ лин. зависни $\lightning$ контрадикција

$\boxed{\Leftarrow}$ Нека је $a\neq b\neq c\neq a$. Доказ. $\vec{a},\vec{b}$ и $\vec{c}$ лин. независни

$$x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$$

$$(x,ax,a^2x)+(y,by,b^2y)+(z,cz,c^2z)=0$$

$$\begin{aligned} \boxed{x}+y+z&=0 \quad \cdot(-a)\ \cdot(-a^2)\\ ax+by+cz&=0 \quad +\\ a^2x+b^2y+c^2z&=0 \quad + \end{aligned}$$

$$\begin{aligned} x+y+z&=0\\ \boxed{(b-a)y}+(c-a)z&=0 \quad \cdot(-(b+a))\\ (b^2-a^2)y+(c^2-a^2)z&=0 \quad + \end{aligned}$$

$$\begin{aligned} x+y+z&=0\\ (b-a)y+(c-a)z&=0\\ (c-a)[-b-a+c+a]z&=0 \end{aligned}$$

$$\Rightarrow \begin{aligned} \text{I)}\quad x+y+z&=0\\ \text{II)}\quad (b-a)y+(c-a)z&=0\\ \text{III)}\quad (c-a)(c-b)z&=0 \end{aligned}$$

III) Како је $c\neq a$, $c\neq b$ $\Rightarrow \boxed{z=0}$

II) $b\neq a \Rightarrow \boxed{y=0}$

I) $\boxed{x=0}$

$\Rightarrow$ $\vec{a},\vec{b}$ и $\vec{c}$ су линеарно независни ∎