СЕПТ 2 ЛААГ

1 а) Вект. $v_1, \dots, v_n$ су линеарно независни ако

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

б) $\mathrm{Span}\{v_1, \dots, v_k\} = \mathcal{L}\{v_1, \dots, v_k\} = \{\alpha_1 v_1 + \dots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$

в) Траг матрице је збир елемената на главној дијагонали.

$$tr\left(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix}\right) = a_{11} + a_{22} + \dots + a_{nn}.$$

г) $S^\perp = \{v \in V \mid \langle v, s \rangle = 0, \forall s \in S\} \le V$

д) Грасм. ф-ла: $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

ђ) Бине-Коши: $\det(A \cdot B) = \det A \cdot \det B$

е) $[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 5 & 2 \end{vmatrix} \begin{matrix} 1 & 2 \\ 1 & 1 \\ -1 & 5 \end{matrix} = \begin{matrix} 2 - 2 + 15 \\ +3 - 5 - 4 \end{matrix} = 9 \Rightarrow V = |9| = \boxed{9}$

ж) A и B сличне $\Leftrightarrow (\exists P)\ A = P^{-1} B P$

$$\overset{\varphi_A(\lambda)}{\det(A - \lambda E)} = \det(P^{-1} B P - \lambda E) = \det(P^{-1}(B - \lambda E) P) =$$
$$= \cancel{\det P^{-1}} \cdot \det(B - \lambda E) \cdot \cancel{\det P} = \det(B - \lambda E) = \varphi_B(\lambda)$$

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2 $L : M_2(\mathbb{R}) \to \mathbb{R}^2, \quad L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b - c + d,\ 2a + b + c - d)$

а) I) $L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right)$

$= (a_1 + a_2 - 2(b_1 + b_2) - (c_1 + c_2) + d_1 + d_2,\ 2(a_1 + a_2) + b_1 + b_2 + c_1 + c_2 - d_1 - d_2)$

$= (a_1 - 2b_1 - c_1 + d_1,\ 2a_1 + b_1 + c_1 - d_1) + (a_2 - 2b_2 - c_2 + d_2,\ 2a_2 + b_2 + c_2 - d_2)$

$= L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

II) $L\left(\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right) = (\alpha a + 2\alpha b - \alpha c + \alpha d,\ 2\alpha a + \alpha b + \alpha c - \alpha d) = \alpha L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

I и II $\Rightarrow$ L је линеарно

б) I) $\mathrm{Ker}\, L = \{A \in M_2(\mathbb{R}) \mid L(A) = 0\} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \,\middle|\, \begin{matrix} a + 2b - c + d = 0 \\ 2a + b + c - d = 0 \end{matrix}\right\}$

$$\begin{aligned} a + 2b - c + d &= 0 \\ 2a + b + c - d &= 0 \quad (+) \\ \hline 3a + 3b = 0 &\Rightarrow b = -a \\ a + c - d = 0 &\Rightarrow d = a + c \end{aligned}$$

$= \left\{\begin{bmatrix} a & -a \\ c & a + c \end{bmatrix} \,\middle|\, a, c \in \mathbb{R}\right\} = \mathcal{L}\left\{\underset{e_1}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}\right\}$

$\boxed{e_1, e_2 \text{ лин. нез} \Rightarrow \text{база за } \mathrm{Ker}\, L}$

II) $\mathrm{Im}\, L = \mathcal{L}\{L(E_1), L(E_2), L(E_3), L(E_4)\} = \mathcal{L}\{\overset{f_1}{(1,2)}, \overset{f_2}{(2,1)}, \overset{f_3}{(-1,1)}, \overset{f_4}{(1,-1)}\}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{matrix} (-2) & (-1) \\ \\ + \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & 3 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} f_1 = (1,2) \\ f_2 = (2,1) \end{matrix} \text{ база за } \mathrm{Im}\, L}$$

3 а)

$$\varphi_A(\lambda) = \det(A-\lambda E) = \begin{vmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2-\lambda & 0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -2 & -3 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda)(1-\lambda) \Rightarrow \boxed{\begin{array}{l} \varphi_A(\lambda) = -(\lambda-1)(\lambda-2)(\lambda-3) \\ \mu_A(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3) \end{array}}$$

$\boxed{\lambda_1 = 1}$

$$\begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{aligned} a + 2b + 3c &= 0 \\ -a \quad\;\; - c &= 0 \Rightarrow a = -c \\ 2a + 2b + 4c &= 0 \\ \hline 2b + 2c &= 0 \\ 2b + 2c &= 0 \end{aligned} \Rightarrow b = -c$$

$$\Rightarrow \boxed{v_1 = c\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}$$

$\boxed{\lambda_2 = 2}$

$$\begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{aligned} 2a + 2b + 3c &= 0 \\ -a - b - c &= 0 \quad \cdot 2 \\ \hline c &= 0 \\ b &= -a \end{aligned}$$

$$\boxed{v_2 = a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$$

$\boxed{\lambda_3 = 3}$

$$\begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{aligned} 3a + 2b + 3c &= 0 \\ -a - 2b - c &= 0 \quad \cdot 3 \\ 2a + 2b + 2c &= 0 \\ \hline -4b &= 0 \Rightarrow b = 0 \\ a &= -c \end{aligned}$$

$$\boxed{v_3 = c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

б) A јесте дијагоналног типа јер $\mu_A(\lambda)$ нема вишеструке нуле

$$\Rightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

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4 $\quad A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

а) $\langle\, , \rangle : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$, $\langle X, Y\rangle = tr(X^T A Y)$ је ск. производ?

I) $\langle \alpha X + \beta Y, Z\rangle = tr((\alpha X + \beta Y)^T A Z) = tr((\alpha X^T + \beta Y^T) A Z)$

$= tr(\alpha X^T A Z + \beta Y^T A Z) = \alpha\, tr(X^T A Z) + \beta\, tr(Y^T A Z)$

$= \alpha \langle X, Z\rangle + \beta\langle Y, Z\rangle$

користимо $tr(\alpha A) = \alpha \cdot tr A$, $tr(A+B) = tr(A) + tr(B)$ и $tr(A^T) = tr A$.

II)

$$\langle X, Y\rangle = tr(X^T A Y) = tr(X^T A Y)^T = tr(Y^T A^T X) \overset{A^T = A}{=} tr(Y^T A X) = \langle Y, X\rangle$$

III) Нека је $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Rightarrow \langle X, X\rangle = tr(X^T A X) = tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = tr\left(\begin{bmatrix} 2a & 5c \\ 2b & 5d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 2a^2 + 5c^2 & \sim \\ \sim & 2b^2 + 5d^2 \end{bmatrix}\right) = 2a^2 + 5c^2 + 2b^2 + 5d^2 \geq 0$$

$$\langle X, X\rangle = 0 \Leftrightarrow a = b = c = d = 0 \Leftrightarrow X = 0$$

I, II, III $\Rightarrow$ $\langle\, , \rangle$ је скаларни производ на $M_2(\mathbb{R})$

б) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} 2a & 5b \\ 2c & 5d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 5c & 5d \end{bmatrix} \right\}$

$\Rightarrow b = 0, c = 0 \quad = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} = \mathcal{L}\left(\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4}\right)$

$I = I_1 + I_2$, $I_1 \in W$, $I_2 \in W^\perp$

$I = \alpha E_1 + \beta E_4 + I_2$, очигледно $\alpha = 1$, $\beta = 1$

$\Rightarrow I = E_1 + E_4 \in W \Rightarrow$ пројекција I на W је I, на $W^\perp$ је 0 ■

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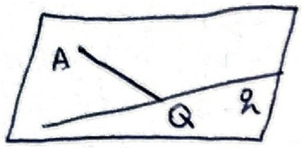
5 $A(1,2,3)$, $q: \dfrac{x-1}{1} = \dfrac{y-2}{1} = \dfrac{z}{1}$

а)



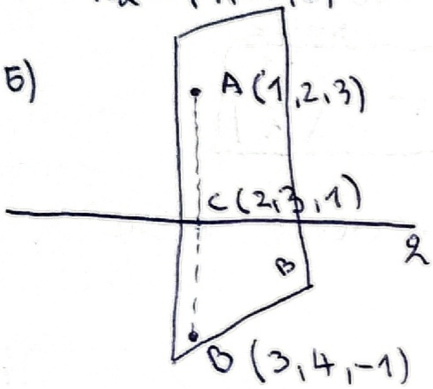
$A(1,2,3) \in \alpha$, $\vec{q} = (1,1,1)$, $Q(1,2,0) \in q$

$\left.\begin{array}{l} \vec{AQ} = (0,0,-3) \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{AQ} \\ q \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{q} \end{array}\right\} \Rightarrow \vec{n}_\alpha = \vec{AQ} \times \vec{q} = \begin{vmatrix} i & j & k \\ 0 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = (3,-3,0)$

$\Rightarrow A(1,2,3) \in \alpha$, $\vec{n}_\alpha = (1,-1,0) \Rightarrow \alpha: 1(x-1) - 1(y-2) + 0(z-3) = 0$

$\boxed{\alpha: x - y + 1 = 0}$

б)



I) β т.д. $A \in \beta$, $\vec{n}_\beta = \vec{q}$ $\Rightarrow \beta: x-1+y-2+z-3 = 0$

$x + y + z - 6 = 0$

II) $\beta \cap q = C \Rightarrow t+1+t+2+t-6 = 0$

$3t - 3 = 0 \Rightarrow t = 1$

$\Rightarrow C(2,3,1)$

$\Rightarrow \boxed{B(3,4,-1)}$ ■

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6 а) $\det(A) = \det(A^T) \overset{A^T=-A}{=} \det(-A) = (-1)^n \det A \overset{n \text{ непаран}}{=} -\det A \Rightarrow \det A = 0$

б) $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A \Rightarrow \operatorname{adj} A = A^{-1} \cdot \det A$

$\Rightarrow \det(\operatorname{adj} A) = \det(\det A \cdot A^{-1}) \overset{A \in M_n(\mathbb{R})}{=} (\det A)^n \cdot \underbrace{\det(A^{-1})}_{= \frac{1}{\det A}} = (\det A)^{n-1}$ ■