ЈАНУАР 2

1 а) База је линеарно независан генераторни скуп вектора V. Димензија простора је број елемената базе.

Стандардне базе: за $\mathbb{R}^3$ је $e_1=(1,0,0)$, $e_2=(0,1,0)$ и $e_3=(0,0,1)$

за $M_2(\mathbb{R})$ је $E_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $E_2=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $E_3=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ и $E_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}$.

за $\mathbb{R}^3[x]$ је $p_1=1$, $p_2=x$, $p_3=x^2$ и $p_4=x^3$.

б) $W=\mathrm{Span}(v_1,\dots v_k)=\{\alpha_1v_1+\alpha_2v_2+\dots+\alpha_kv_k \mid \alpha_i\in\mathbb{R}\}\le V$

1°) $0=0\cdot v_1+0v_2+\dots+0\cdot v_k\in \mathrm{Span}(v_1,\dots v_k)=W$

2°) $u=\alpha_1v_1+\dots+\alpha_kv_k$, $v=\beta_1v_1+\dots+\beta_kv_k\in W \Rightarrow u+v\in W$?

$u+v=(\alpha_1+\beta_1)v_1+(\alpha_2+\beta_2)v_2+\dots+(\alpha_k+\beta_k)v_k\in W$

$\Rightarrow \boxed{W\le V}$

в) A и B сличне матрице $\Rightarrow (\exists P)\ A=PBP^{-1}$

$\Rightarrow \det A=\det(PBP^{-1})=\det P\cdot\det B\cdot\det P^{-1}=\det B$

г) $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in M_2(\mathbb{R})$

$\varphi_A(\lambda)=\begin{vmatrix}a-\lambda&b\\c&d-\lambda\end{vmatrix}=(a-\lambda)(d-\lambda)-bc$

$=\lambda^2-(a+d)\lambda+ad-bc=\lambda^2-\lambda\,\mathrm{tr}A+\det A$

д)

$\|u+v\|^2=\langle u+v,u+v\rangle=\langle u,u\rangle+2\langle u,v\rangle+\langle v,v\rangle$

$\|u-v\|^2=\langle u-v,u-v\rangle=\langle u,u\rangle-2\langle u,v\rangle+\langle v,v\rangle$ +

$\|u+v\|^2+\|u-v\|^2=2\|u\|^2+2\|v\|^2 \Rightarrow \|u\|^2+\|v\|^2=\frac{1}{2}(\|u+v\|^2+\|u-v\|^2)$

2 $L:\mathbb{R}^3\to\mathbb{R}^3$, $L(x,y,z)=(x-y+z,\ x-3z,\ 2x-y-2z)$

a) $[L]_S$, $S=\{f_1=(1,1,0),\ f_2=(1,2,3),\ f_3=(1,3,5)\}$

$(a,b,c)_E=xf_1+yf_2+zf_3=(x,y,z)_S$

$$\begin{aligned}x+y+z&=a \quad /\cdot(-1)\\ x+2y+3z&=b\\ 3y+5z&=c\end{aligned}\Rightarrow \begin{aligned}x+y+z&=a\\ y+2z&=b-a \quad /\cdot(-3)\\ 3y+5z&=c\end{aligned}\Rightarrow \begin{aligned}x&=-a+2b-c\\ y&=5a-5b+2c\\ z&=-3a+3b-c\end{aligned}$$

$\Rightarrow \boxed{(a,b,c)_E=(-a+2b-c,\ 5a-5b+2c,\ -3a+3b-c)_S}$

$L(f_1)=(0,1,1)_E=(1,-3,2)_S$

$L(f_2)=(2,-8,-6)_E=(-12,38,-24)_S$

$L(f_3)=(3,-14,-11)_E=(-20,63,-40)_S$

$\Rightarrow [L]_S=\begin{bmatrix}1&-12&-20\\-3&38&63\\2&-24&-40\end{bmatrix}$

б) БАЗА ЗА $\operatorname{Im} L$:

$\boxed{\text{I НАЧИН}}$ КАКО JE f_1, f_2, f_3 БАЗА ЗА $\mathbb{R}^3$

$\Rightarrow \operatorname{Im} L = \operatorname{Span}\{L(f_1), L(f_2), L(f_3)\} = \operatorname{Span}\{(0,1,1), (2,-8,-6), (3,-14,-11)\}$

$\boxed{\text{II НАЧИН}}$ СТАНДАРДНА БАЗА (e_1, e_2, e_3) ЗА $\mathbb{R}^3$

$\Rightarrow L(e_1) = (1,1,2)$, $L(e_2) = (-1,0,-1)$, $L(e_3) = (1,-3,-2)$

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 1 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ БАЗА ЗА $\operatorname{Im} L$ JE $\{(1,1,2), (-1,0,-1)\}$, $\rho(L) = 2$

$\operatorname{Ker} L = \{(x,y,z) \mid L(x,y,z) = 0\} =$

$$\begin{aligned} x - y + z &= 0 \\ x - 3z &= 0 \\ 2x - y - 2z &= 0 \end{aligned} \Rightarrow \begin{aligned} x - y + z &= 0 \\ x - 3z &= 0 \\ x - 3z &= 0 \end{aligned} \Rightarrow \begin{aligned} x - y + z &= 0 \\ x - 3z &= 0 \end{aligned} \Rightarrow \begin{aligned} x &= 3\alpha \\ y &= 4\alpha \\ z &= \alpha, \ \alpha \in \mathbb{R} \end{aligned}$$

$\Rightarrow \boxed{\operatorname{Ker} L = \mathcal{L}((3,4,1)) \Rightarrow \delta(L) = 1}$ $\rho(L) + \delta(L) = 3 = \dim \mathbb{R}^3$

$\boxed{3}$

$$\begin{aligned} x \quad\quad + 3z &= 1 \\ -3x + 2y - (8+a)z &= 2-a \\ 2x + (a-4)y + z &= 1 \end{aligned}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 3 \\ -3 & 2 & -(8+a) \\ 2 & a-4 & 1 \end{vmatrix} = \ldots = a^2 - 5a - 6 = \boxed{(a-6)(a+1)}$$

$$\Delta_x = \begin{vmatrix} 1 & 0 & 3 \\ 2-a & 2 & -(8+a) \\ 1 & a-4 & 1 \end{vmatrix} = \ldots = -2(a^2 - 11a + 30) = \boxed{-2(a-6)(a-5)}$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 3 \\ -3 & 2-a & -(8+a) \\ 2 & 1 & 1 \end{vmatrix} = \ldots = 4a - 24 = \boxed{4(a-6)}$$

$$\Delta_z = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 2 & 2-a \\ 2 & a-4 & 1 \end{vmatrix} = \ldots = a^2 - 9a + 18 = \boxed{(a-6)(a-3)}$$

1) $a \neq 6, a \neq -1 \Rightarrow \Delta \neq 0 \Rightarrow$ ЈЕДИНСТВЕНО РЕШЕЊЕ

$$\boxed{(x,y,z) = \left(\frac{\Delta x}{\Delta}, \frac{\Delta y}{\Delta}, \frac{\Delta z}{\Delta}\right) = \left(\frac{-2(a-5)}{a+1}, \frac{4}{a+1}, \frac{a-3}{a+1}\right)}$$

$a \neq 6, a \neq -1$

2) $a=-1 \Rightarrow \Delta = 0$ али $\Delta_Y = -28 \neq 0$

$\Rightarrow$ $\boxed{\text{НЕМА РЕШЕЊА}}$

3) $a=6 \Rightarrow \Delta = 0 = \Delta_X = \Delta_Y = \Delta_Z \Rightarrow$ не може Крамер

$$\Rightarrow \begin{array}{l} x \quad + 3z = 1 \\ -3x+2y-14z=-4 \quad \cdot(-1) \\ 2x+\boxed{2y}+z=1 \end{array} \Rightarrow \begin{array}{l} 2x+2y+z=1 \\ x \quad +3z=1 \quad \cdot(-5) \\ 5x \quad +15z=5 \end{array} \quad \begin{array}{l} 2x+2y+z=1 \\ x \quad +3z=1 \\ 0 \quad =0 \end{array}$$

$\Rightarrow z = \alpha, \ \alpha \in \mathbb{R}$

$x = 1-3\alpha$

$y = \frac{1}{2}(1-\alpha-2+3\alpha) = \alpha - \frac{1}{2}$

$$\Rightarrow \boxed{\mathcal{R}_\alpha = \{(x,y,z) = (1-3\alpha, \alpha-\tfrac{1}{2}, \alpha) \mid \alpha \in \mathbb{R}\}, \ a=6}$$

$\boxed{4}$ $W = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1+2x_2+x_3=0\}$

a) $x_1+2x_2+x_3=0 \Rightarrow \begin{array}{l} x_2=\alpha \\ x_3=\beta, \ \alpha,\beta\in\mathbb{R} \\ x_4 = \gamma, \ \gamma\in\mathbb{R} \end{array} \Rightarrow x_1 = -2\alpha-\beta$

$\Rightarrow W = \{(-2\alpha-\beta, \alpha, \beta, \gamma) \mid \alpha,\beta,\gamma \in \mathbb{R}\} = \mathcal{L}(\overset{f_3}{(-2,1,0,0)}, \overset{f_2}{(-1,0,1,0)}, \overset{f_1}{(0,0,0,1)})$

$e_1 = f_1 = (0,0,0,1)$

$$e_2 = f_2 - \frac{f_2 \circ e_1}{e_1 \circ e_1} e_1 = (-1,0,1,0) - \cancel{\frac{(-1,0,1,0)\circ(0,0,0,1)}{(0,0,0,1)\circ(0,0,0,1)}}^{\,0} e_1 = (-1,0,1,0)$$

$$e_3 = f_3 - \cancel{\frac{f_3 \circ e_1}{e_1\circ e_1}}^{\,0} e_1 - \frac{f_3\circ e_2}{e_2 \circ e_2} e_2 = (-2,1,0,0) - \frac{2}{2}(-1,0,1,0) = (-1,1,-1,0)$$

$$\Rightarrow \hat{e}_1 = \frac{e_1}{\|e_1\|} = e_1 = \boxed{(0,0,0,1)}$$

$$\hat{e}_2 = \frac{e_2}{\|e_2\|} = \frac{(-1,0,1,0)}{\sqrt{2}} = \boxed{\left(-\tfrac{\sqrt{2}}{2}, 0, \tfrac{\sqrt{2}}{2}, 0\right)}$$

$$\hat{e}_3 = \frac{e_3}{\|e_3\|} = \frac{(-1,1,-1,0)}{\sqrt{3}} = \boxed{\left(-\tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}, -\tfrac{\sqrt{3}}{3}, 0\right)}$$

б) $W = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid (1,2,1,0)\circ(x_1,x_2,x_3,x_4) = 0\} = \mathcal{L}(\overset{g_1}{(1,2,1,0)})$

в) $v = (1,2,3,4) = v_1 + v_2, \quad v_1 \in W, \quad v_2 \in W^\perp$

$v = v_1 + \alpha \cdot g_1 \quad / \circ g_1 \Rightarrow \alpha = \frac{v \circ g_1}{g_1 \circ g_1} = \frac{8}{6} = \frac{4}{3}$

$\Rightarrow v_2 = \frac{4}{3}(1,2,1,0) = \left(\frac{4}{3}, \frac{8}{3}, \frac{4}{3}, 0\right)$

$v_1 = v - v_2 = (1,2,3,4) - \left(\frac{4}{3}, \frac{8}{3}, \frac{4}{3}, 0\right) \Rightarrow \boxed{v_1 = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{5}{3}, 4\right)}$

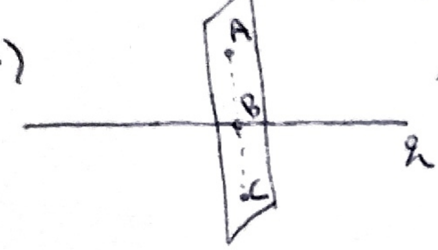
$$\cos\measuredangle(u,w)=\cos\measuredangle(u,u_1)=\frac{u\circ u_1}{\|u\|\cdot\|u_1\|}=\frac{(1,2,3,4)\circ\left(-\frac{1}{3},-\frac{2}{3},\frac{5}{3},4\right)}{\sqrt{1+4+9+16}\sqrt{\frac{1}{9}+\frac{4}{9}+\frac{25}{9}+\frac{144}{9}}}$$

$$\cos\measuredangle(u,w)=\frac{\frac{58}{3}}{\sqrt{30}\cdot\frac{\sqrt{174}}{3}}\Rightarrow \boxed{\measuredangle(u,w)=\arccos\frac{58}{\sqrt{30}\sqrt{174}}}$$

$$d(u,w)=\|u_2\|=\sqrt{\frac{16}{9}+\frac{64}{9}+\frac{16}{9}}=\boxed{\frac{\sqrt{96}}{3}}$$

5 $A(2,-3,1)$, $p:\frac{x+1}{3}=\frac{y+2}{-3}=\frac{z-1}{2}$, $q:\frac{x-7}{2}=\frac{y+4}{-2}=\frac{z-1}{2}$

a)



1) α т.д. $A\in\alpha$, $\alpha\perp q$

$\Rightarrow \alpha: 2(x-2)-2(y+3)+2(z-1)=0$

$\alpha: 2x-2y+2z-12=0$

$\boxed{\alpha: x-y+z-6=0}$

$B=q\cap\alpha$: $q:\begin{cases}x=2t+7\\ y=-2t-4\\ z=2t+1\end{cases}\Rightarrow 2t+7+2t+4+2t+1-6=0$

$6t+6=0\Rightarrow t=-1$

$\Rightarrow A(2,-3,1)$, $B(5,-2,-1)\Rightarrow \boxed{C(8,-1,-3)}$

Б) $P(-1,-2,1)$, $\vec{p}=(3,-3,2)$

$Q(7,-4,1)$, $\vec{q}=(2,-2,2)$, $e:\frac{x-2}{a}=\frac{y+3}{b}=\frac{z-1}{c}$

1°) $[\vec{p},\vec{e},\vec{PA}]=0=\left|\begin{array}{ccc|cc}3&-3&2&3&-3\\a&b&c&a&b\\3&-1&0&3&-1\end{array}\right|=-9c-2a-6b+3c$

$\Rightarrow \boxed{-a-3b-3c=0}$

2°) $[\vec{q},\vec{e},\vec{QA}]=0=\left|\begin{array}{ccc|cc}2&-2&2&2&-2\\a&b&c&a&b\\-5&1&0&-5&1\end{array}\right|=10c+2a+10b-2c$

$\Rightarrow \boxed{a+5b+4c=0}$ + $\Rightarrow 2b+c=0\Rightarrow \boxed{c=-2b}$

$\Rightarrow a=-5b+8b=3b\Rightarrow \vec{e}=b(3,1,-2)$

$$\Rightarrow \boxed{e:\frac{x-2}{3}=\frac{y+3}{1}=\frac{z-1}{-2}}$$

$\boxed{6}$ $U \neq W$, $U, W \leq V$, $\dim U = \dim W = 5$, $\dim V = 7$.

* $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$\Rightarrow \dim(U \cap W) = \underset{5}{\underset{\|}{\dim U}} + \underset{5}{\underset{\|}{\dim W}} - \underset{\leq 7}{\underbrace{\dim(U+W)}} \Rightarrow \boxed{\dim(U \cap W) \geq 3}$

* $U \cap W \leq U$, $U \cap W \leq V$ $\Rightarrow \dim(U \cap W) \leq 5 \Rightarrow 3 \leq \dim(U \cap W) \leq 5$

* Ako je $\dim(U \cap W) = 5$

$\left.\begin{array}{l} U \cap W \leq U \\ U \cap W \leq W \\ \dim U = \dim W = 5 \end{array}\right\} \Rightarrow U = W$ ↯ $\Rightarrow \dim(U \cap W) \neq 5$

1) $\dim(U \cap W) = 3$

$V = \mathbb{R}^7$

$U = \mathcal{L}(e_1, e_2, e_3, e_4, e_5)$

$W = \mathcal{L}(e_1, e_2, e_3, e_6, e_7)$

$U \cap W = \mathcal{L}(e_1, e_2, e_3)$

$U + W = \mathbb{R}^7$

2) $\dim(U \cap W) = 4$

$V = \mathbb{R}^7$

$U = \mathcal{L}(e_1, e_2, e_3, e_4, e_5)$

$W = \mathcal{L}(e_1, e_2, e_3, e_4, e_6)$

$U \cap W = \mathcal{L}(e_1, e_2, e_3, e_4)$

$U + W = \mathcal{L}(e_1, e_2, e_3, e_4, e_5, e_6)$ ■