ЈАНУАР 1

1 а) $V = U + W = \{u + w \mid u \in U, w \in W\}$

$V = U \oplus W$ ако сваки вектор из V може на јединствен начин да се запише у облику $u + w$ $\Leftrightarrow$ $V = U + W$ и $U \cap W = \emptyset$.

б) $L : U \to V$

$\operatorname{Im} L = \{v \in V \mid v = L(u)$ за неко $u \in U\}$, $\rho(L) = \dim \operatorname{Im} L$

$\operatorname{Ker} L = \{u \in U \mid L(u) = 0\}$, $\delta(L) = \dim \operatorname{Ker} L$

в) $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \Delta x_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix}, \Delta x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{22} & b_3 & a_{33} \end{vmatrix}, \Delta x_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Т 1) $\Delta \neq 0 \Rightarrow$ јединствено решење $x_i = \frac{\Delta x_i}{\Delta}$, $i = 1,2,3$

2) $\Delta = 0$ али $\Delta x_i \neq 0$ за неко $i = 1,2,3 \Rightarrow$ нема решења

3) $\Delta = 0$, $\Delta x_i = 0$, $i = 1,2,3 \Rightarrow$ не даје одговор

г) $\langle , \rangle : V \times V \to \mathbb{R}$ је скаларни производ ако важи

1) линеарност по 1. аргументу

$\langle au + bv, w \rangle = a\langle u, w\rangle + b\langle v, w\rangle$, $a, b \in \mathbb{R}$, $u, v, w \in V$

2) комутативност $\langle u, v\rangle = \langle v, u\rangle$

3) позитивна дефинитност: $(\forall u) \langle u, u\rangle \geq 0$, $\langle u, u\rangle = 0 \Leftrightarrow u = 0$

д) $\{u_1, u_2, \ldots, u_n\}$ ортонормиран скуп

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$ $/\cdot u_1$ $/\cdot u_2$

$\Rightarrow \alpha_1 \cdot \|u_1\|^2 = 0 \Rightarrow \alpha_1 = 0$

$\Rightarrow \alpha_2 = 0 \ldots, \alpha_n = 0 \Rightarrow$ линеарно независни

ђ) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ је вектор т.д.

1) правац: $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$

2) смер: $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ поз. орј.

3) $\|\vec{a} \times \vec{b}\| = P_{\square} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin \alpha$

$$[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \circ \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\vec{AB} = (1,1,1)$, $\vec{AC} = (-1,0,1)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = (1, -2, 1)$$

$$P_\Delta = \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{1+4+1} = \boxed{\frac{\sqrt{6}}{2}}$$

$\boxed{2}$ Посматрамо станд. базу за $\mathbb{R}^3[t]$: $t^3, t^2, t, 1$

$$\Rightarrow U = \mathcal{L}\{\underset{p_1}{(1,2,3,1)}, \underset{p_2}{(3,5,7,0)}, \underset{p_3}{(1,1,1,-2)}, \underset{p_4}{(1,0,-2,-8)}\}$$

$$V = \mathcal{L}\{\underset{q_1}{(1,1,1,-2)}, \underset{q_2}{(1,2,2,-1)}, \underset{q_3}{(2,5,5,-1)}\}$$

dim U:

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array}\begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 3 & 5 & 7 & 0 \\ 1 & 1 & 1 & -2 \\ 1 & 0 & -2 & -8 \end{bmatrix} \begin{matrix} -3 & -1 & -1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -5 & -9 \end{bmatrix} \begin{matrix} -1 & -2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$\Rightarrow$ $\dim U = 3$, база за U је $\{p_1, p_2, p_4\}$

dim V:

$$\begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array}\begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 2 & 2 & -1 \\ 2 & 5 & 5 & -1 \end{bmatrix} \begin{matrix} -1 & -2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix} \begin{matrix} -3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $\dim V = 2$, база за V је $\{q_1, q_2\}$

U+V:

$$\begin{array}{c} p_1 \\ p_2 \\ p_4 \\ q_1 \\ q_2 \end{array}\begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & 5 & 7 & 0 \\ 1 & 0 & -2 & -8 \\ 1 & 1 & 1 & -2 \\ 1 & 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -5 & -9 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \dim(U+V) = 4$$

Грасман: $\dim(U+V) = \dim U + \dim V - \dim U \cap V$

$$4 = 3 + 2 - \dim U\cap V \Rightarrow \boxed{\dim U \cap V = 1}$$

$\boxed{3}$ $L(x,y,z) = (x+y+2z,\ x+z,\ 2x+y+2z)$

а) 1) $L((x_1+x_2, y_1+y_2, z_1+z_2)) = (x_1+x_2+y_1+y_2+2(z_1+z_2),\ x_1+x_2+z_1+z_2,$
$2x_1+2x_2+y_1+y_2+2z_1+2z_2) = L(x_1,y_1,z_1) + L(x_2,y_2,z_2)$

2) $L((kx,ky,kz)) = kL(x,y,z)$ $\Rightarrow$ јесте линеарно.

б) $L(1,0,0) = (1,1,2)$

$L(0,1,0) = (1,0,1)$ $\Rightarrow$ $[L]_e = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$L(0,0,1) = (2,1,2)$

в) $\operatorname{Ker} L = \{(a,b,c) \mid L(a,b,c) = 0\}$

$= \{(a,b,c) \mid (a+b+2c,\ a+c,\ 2a+b+2c) = 0\}$

$$\left.\begin{aligned} a+b+2c&=0 \quad \cdot(-1)\\ a+c&=0\\ 2a+b+2c&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} a+b+2c&=0\\ a+c&=0\\ a&=0 \end{aligned} \Rightarrow b=0,\ c=0 \Rightarrow \operatorname{Ker} L = \{0\}$$

$\Rightarrow L$ инвертибилан

$L^{-1}(L(x,y,z)) = (x,y,z)$

означимо $L(x,y,z) = (a,b,c) \Rightarrow L^{-1}(a,b,c) = (x,y,z)$

$L(x,y,z) = (x+y+2z,\ x+z,\ 2x+y+2z)$

$$\Rightarrow \left.\begin{aligned} x+y+2z&=a \quad \cdot(-1)\\ x+z&=b\\ 2x+y+2z&=c \end{aligned}\right\} \Rightarrow \begin{aligned} x+y+2z&=a\\ x+z&=b\\ x&=c-a \end{aligned} \quad \begin{aligned} &\Rightarrow y = a+a-c-2a-2b+2c\\ &\Rightarrow z = a+b-c \end{aligned}$$

$$\Rightarrow \boxed{L^{-1}(a,b,c) = (c-a,\ -2b+c,\ a+b-c)}$$

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4. $A = \begin{bmatrix} 3 & 0 & -2 \\ -9 & -1 & 9 \\ 1 & 0 & 0 \end{bmatrix}$

а) $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & -2 \\ -9 & -1-\lambda & 9 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix}$

$= -(\lambda+1)(-3\lambda+\lambda^2+2) = \boxed{-(\lambda+1)(\lambda-1)(\lambda-2) = \varphi_A(\lambda)}$

минимални пол. садржи исте нуле $\Rightarrow \boxed{\mu_A(\lambda) = (\lambda+1)(\lambda-1)(\lambda-2)}$

б) $\varphi_A(\lambda) = 0 \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 1 \quad \lambda_3 = 2$

1) $(A+I)v = 0$

$\begin{bmatrix} 4 & 0 & -2 \\ -9 & 0 & 9 \\ 1 & 0 & 1 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$\Rightarrow \begin{aligned} 4a-2c&=0\\ -9a+9c&=0\\ a+c&=0 \end{aligned} \Rightarrow \begin{aligned} a&=0\\ c&=0 \end{aligned}$

$\Rightarrow \boxed{v_1 = b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

2) $(A-I)v = 0$

$\begin{bmatrix} 2 & 0 & -2 \\ -9 & -2 & 9 \\ 1 & 0 & -1 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$\Rightarrow \begin{aligned} 2a-2c&=0\\ -9a-2b+9c&=0\\ a-c&=0 \end{aligned} \Rightarrow \begin{aligned} a&=c\\ b&=0 \end{aligned}$

$\Rightarrow \boxed{v_2 = a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

3) $(A-2I)v = 0$

$\begin{bmatrix} 1 & 0 & -2 \\ -9 & -3 & 9 \\ 1 & 0 & -2 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$\Rightarrow a = 2c$

$3b = -9a+9c = -9c$

$\boxed{v_3 = c\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}$

в) три лин. независна сопствена вектора (минимални полином нема вишеструке нуле) $\Rightarrow A$ је дијагоналног типа

$$\Rightarrow D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

г) $P^{-1} = \dots = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}$, $D = P^{-1}AP$

$\Rightarrow A = PDP^{-1} \Rightarrow A^n = PD^nP^{-1}$

$$A^n = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2^{n+1} \\ (-1)^n & 0 & -3\cdot 2^n \\ 0 & 1 & 2^n \end{bmatrix} \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} -1+2^{n+1} & 0 & 2-2^{n+1} \\ 3(-1)^n-3\cdot 2^n & (-1)^n & -3(-1)^n+3\cdot 2^n \\ -1+2^n & 0 & 2-2^n \end{bmatrix}}$$

$\boxed{5}$ а) $q: x+y=5 \quad \cdot 2$
$r: 3x-2y=5 \quad +$

$5x = 15 \Rightarrow x=3$
$y=2$

$\Rightarrow q \cap r = A(3,2)$ $\quad x: y=0 \Rightarrow \vec{n}_x = (0,1) = \vec{p}$

$A(3,2) \in p$, $\vec{p} = (0,1)$ $\Rightarrow$ $p: \frac{x-3}{0} = \frac{y-2}{1}$

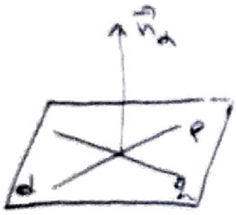
$\boxed{p: \ x=3}$

б) $p: \begin{array}{l} x=-t+3 \\ y=2t-2 \\ z=t+1 \end{array}, \ t\in\mathbb{R}$ $\qquad q: \begin{array}{l} x=s \\ y=3s-1 \\ z=5s-2 \end{array}, \ s\in\mathbb{R}$

пресек:
$-t+3=s$
$2t-2=3s-1$
$t+1=5s-2$

$+ \quad 4=6s-2$
$\Rightarrow s=1$
$t=2$
$4-2=3-1$ ✓

секу се $\Rightarrow \boxed{p\cap q = A(1,2,3)}$



$A(1,2,3)\in\alpha$

$$\vec{n}_\alpha = \vec{q}\times\vec{p} = \begin{vmatrix} i & j & k \\ 1 & 3 & 5 \\ -1 & 2 & 1 \end{vmatrix} = (-7,-6,5)$$

$\Rightarrow \alpha: -7(x-1)-6(y-2)+5(z-3)=0 \Rightarrow \boxed{\alpha: -7x-6y+5z+4=0}$

$\boxed{6}$ а) A регуларна $\Rightarrow A^{-1} = \frac{1}{\det A}\operatorname{adj}A \Rightarrow \operatorname{adj}A = A^{-1}\cdot\det A$

$$\det(\operatorname{adj}A) = \det(\det A\cdot A^{-1}) \overset{A\in M_n}{=} (\det A)^n\cdot\det(A^{-1}) = (\det A)^{n-1}$$

б) A и B регуларне $\Rightarrow A\cdot B^{-1}$ регуларна

$$\det(\operatorname{adj}(AB^{-1})) \overset{a)}{=} (\det(A\cdot B^{-1}))^{2021} = (\det A\cdot\det B^{-1})^{2021}$$

$$= \left(\frac{\det A}{\det B}\right)^{2021} = \left(\frac{20}{21}\right)^{2021}$$