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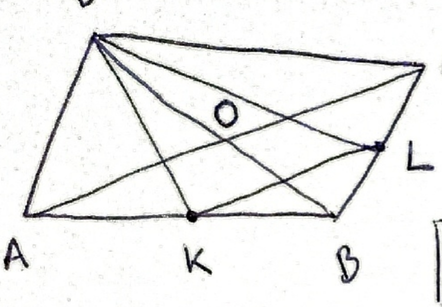

а) O тежиште ΔDKL ?

$\iff \vec{SO} = \frac{1}{3}(\vec{SD} + \vec{SK} + \vec{SL})$ за било које S

Узмимо нпр $S \equiv B$ (или A или C свеједно)

$$\boxed{\frac{1}{3}(\vec{BD} + \vec{BK} + \vec{BL})} = \frac{1}{3}(\vec{BC} + \underbrace{\vec{CD}}_{\vec{BA}} + \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{BC})$$

$$= \frac{1}{3}\left(\frac{3}{2}\vec{BC} + \frac{3}{2}\vec{BA}\right) = \frac{1}{2}\vec{BC} + \frac{1}{2}\vec{BA} = \boxed{\vec{BO}}$$

б) $[O]_{Ae} = [\vec{AO}]_e = (\frac{1}{2}, \frac{1}{2})$

$[\vec{f_1}]_e = [\vec{OK}]_e = (0, -\frac{1}{2})$

$[\vec{f_2}]_e = [\vec{OL}]_e = (\frac{1}{2}, 0)$

$$\Rightarrow \boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$$

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2 $\mathcal{K}: 2x^2 + 2xy + 2y^2 + 4x + 2y - 4 = 0$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

* $\varphi_B(\lambda) = \det(B - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$

* $\lambda_1 = 1$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$a + b = 0 \Rightarrow b = -a$

$\Rightarrow \boxed{\vec{v}_1 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})}$

* $\lambda_2 = 3$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

$-a + b = 0$

$\boxed{\vec{v}_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x &= \frac{1}{\sqrt{2}}(x' + y') \\ y &= \frac{1}{\sqrt{2}}(-x' + y') \end{aligned}}$$

$$\Rightarrow \mathcal{K}': (x'^2 + 2x'y' + y'^2) + (y'^2 - x'^2) + (x'^2 - 2x'y' + y'^2) + 2\sqrt{2}(x' + y') + \sqrt{2}(-x' + y') - 4 = 0$$

$\mathcal{K}': x'^2 + 3y'^2 + \sqrt{2}x' + 3\sqrt{2}y' - 4 = 0$

$\mathcal{K}': (x' + \frac{1}{\sqrt{2}})^2 - \frac{1}{2} + 3(y' + \frac{1}{\sqrt{2}})^2 - \frac{3}{2} - 4 = 0$

$$\Rightarrow \boxed{\begin{aligned} x'' &= x' + \frac{1}{\sqrt{2}} \\ y'' &= y' + \frac{1}{\sqrt{2}} \end{aligned}}$$

$\Rightarrow \mathcal{K}'': x''^2 + 3y''^2 = 6 \iff \boxed{\mathcal{K}'': \frac{x''^2}{6} + \frac{y''^2}{2} = 1} \Rightarrow \boxed{\text{елипса}}$

ТРАНСФОРМАЦИЈА:

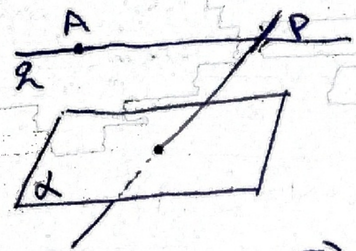
$x = \frac{1}{\sqrt{2}}(x' + y') = \frac{1}{\sqrt{2}}(x'' - \frac{1}{\sqrt{2}} + y'' - \frac{1}{\sqrt{2}})$

$y = \frac{1}{\sqrt{2}}(-x' + y') = \frac{1}{\sqrt{2}}(-x'' + \frac{1}{\sqrt{2}} + y'' - \frac{1}{\sqrt{2}})$

$$\Rightarrow \boxed{\begin{aligned} x &= \frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y'' - 1 \\ y &= -\frac{1}{\sqrt{2}}x'' + \frac{1}{\sqrt{2}}y'' \end{aligned}}$$

3 $A(0,0,7)$, $p: \frac{x+2}{1} = \frac{y+2}{1} = \frac{z-1}{1}$, $\alpha: x+2y+3z+1=0$

a)



$P(t-2, t-2, t+1),\ t\in\mathbb{R} \Rightarrow \underline{q = AP}$

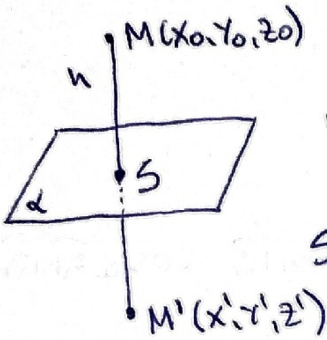
$q \parallel \alpha \Leftrightarrow \vec{q}\circ\vec{n}_\alpha = 0 \Leftrightarrow (t-2, t-2, t-6)\circ(1,2,3)=0$

$\Leftrightarrow t-2+2t-4+3t-18=0 \Leftrightarrow 6t=24 \Leftrightarrow \underline{t=4}$

$\Rightarrow \vec{q} = \vec{AP} = (2,2,-2) \parallel (1,1,-1)$

$A(0,0,7)\in q \Rightarrow \boxed{q: \frac{x}{1} = \frac{y}{1} = \frac{z-7}{-1}}$

б)



$n: \begin{array}{l} x = s + x_0 \\ y = 2s + y_0 \\ z = 3s + z_0 \end{array},\ s\in\mathbb{R}$

$S = \alpha\cap n \Rightarrow s + x_0 + 4s + 2y_0 + 9s + 3z_0 + 1 = 0$

$\Rightarrow s = \frac{-x_0-2y_0-3z_0-1}{14}$

$\Rightarrow S\left(\frac{13x_0-2y_0-3z_0-1}{14}, \frac{-x_0+5y_0-3z_0-1}{7}, \frac{-3x_0-6y_0+5z_0-3}{14}\right)$

$[S] = \frac{[M]+[M']}{2} \Rightarrow [M'] = 2[S]-[M]$

$$\Rightarrow S_\alpha: \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \frac{1}{7}\left(\begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}\right)$$

* слика праве p? $p\cap\alpha: t-2+2t-4+3t+3+1=0 \Rightarrow 6t-2=0$ $t=\frac{1}{3}$

$p\cap\alpha = B\left(-\frac{5}{3}, -\frac{5}{3}, \frac{4}{3}\right)$, $S_\alpha(B)=B$

$\overset{t=2}{\Rightarrow} C(0,0,3)\in p$, $S_\alpha(C) = D\left(-\frac{10}{7}, -\frac{20}{7}, -\frac{9}{7}\right)$

$\boxed{S_\alpha(p) = p' = BD: \frac{x+5/3}{1} = \frac{y+5/3}{-5} = \frac{z-4/3}{-11}}$

$\vec{BD} = \left(\frac{-30+35}{21}, \frac{-60+35}{21}, \frac{-27-28}{21}\right) \parallel (5,-25,-55) \parallel (1,-5,-11)$ ■

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4 $p = \alpha\cap\beta: \begin{array}{l} x+y-z+1=0 \\ 5x-4y+z+2=0 \end{array} \Big]+ \quad \begin{array}{l} x+y-z+1=0 \\ 6x-3y+3=0 \end{array} \Rightarrow p: \begin{array}{l} x=t,\ t\in\mathbb{R} \\ y=2t+1 \\ z=3t+2 \end{array} \Rightarrow \vec{p}=(1,2,3)$

$k: \underbrace{(x-1)^2+(y-1)^2+(z-1)^2=2}_{S}$, $\underbrace{x+y+z=3}_{\varkappa}$ (Како $C\in\varkappa \Rightarrow$ круг k је пресек сфере и равни)

Како је $p\cap\varkappa = (0,1,2)\in k \Rightarrow$ задатак има решење

Нека је $A(x_0,y_0,z_0)\in k$ произв. тачка $\Rightarrow$ изводница цилиндра C је $i: \frac{x-x_0}{1} = \frac{y-y_0}{2} = \frac{z-z_0}{3}$

$\Rightarrow i: \begin{array}{l} x_0 = x-t \\ y_0 = y-2t \\ z_0 = z-3t \end{array}$

1) $A\in\varkappa \Rightarrow x-t+y-2t+z-3t=3$

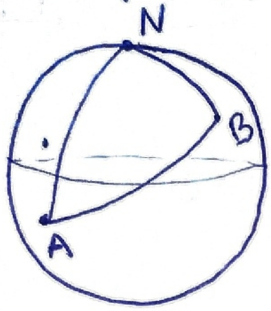
$\Rightarrow t = \frac{x+y+z-3}{6}$

$$\Rightarrow \quad x_0 = \frac{5x-y-z+3}{6}, \quad y_0 = \frac{-x+2y-z+3}{3}, \quad z_0 = \frac{-x-y+z+3}{2}$$

2) $A \in S \Rightarrow \mathcal{C}: \left(\frac{5x-y-z-3}{6}\right)^2 + \left(\frac{-x+2y-z}{3}\right)^2 + \left(\frac{-x-y+z+1}{2}\right)^2 = 2 \;/\cdot 36$

$$\Rightarrow \boxed{\mathcal{C}: (5x-y-z-3)^2 + 4(-x+2y-z)^2 + 9(-x-y+z+1)^2 = 72}$$

$\boxed{5}$ $A(30^\circ S, 135^\circ W)$, $B(60^\circ N, 45^\circ E)$



$A: \varphi_A = -\frac{\pi}{6}, \quad \theta_A = -\frac{3\pi}{4}$

$B: \varphi_B = \frac{\pi}{3}, \quad \theta_B = \frac{\pi}{4}$

$\cos \widehat{AB} = \cos \widehat{NA} \cos \widehat{NB} + \sin \widehat{NA} \sin \widehat{NB} \cos \angle ANB$

$\widehat{NA} = \varphi_N - \varphi_A = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

$\widehat{NB} = \varphi_N - \varphi_B = \pi/2 - \frac{\pi}{3} = \frac{\pi}{6}$

$\angle ANB = |\theta_A - \theta_B| = \left|-\frac{3\pi}{4} - \frac{\pi}{4}\right| = \pi$!

$\Rightarrow$ A, B, N су на великом кругу $\Rightarrow$ не може овако јер немамо сферни троугао.

Најлакше је „по дефиницији", $\widehat{AB} = \angle AOB$ јер је јединична сфера

$$\boxed{\begin{aligned} x &= \cos\varphi \cos\theta \\ y &= \cos\varphi \sin\theta \\ z &= \sin\varphi \end{aligned}} \Rightarrow A\left(\cos -\tfrac{\pi}{6} \cos -\tfrac{3\pi}{4}, \cos -\tfrac{\pi}{6} \sin -\tfrac{3\pi}{4}, \sin\left(-\tfrac{\pi}{6}\right)\right)$$

$$A\left(\frac{\sqrt{3}}{2} \cdot \frac{-\sqrt{2}}{2}, \frac{\sqrt{3}}{2} \cdot \frac{-\sqrt{2}}{2}, -\frac{1}{2}\right) \Rightarrow \boxed{A\left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{4}, -\frac{1}{2}\right)}$$

$$\Rightarrow B\left(\cos\tfrac{\pi}{3} \cos\tfrac{\pi}{4}, \cos\tfrac{\pi}{3} \sin\tfrac{\pi}{4}, \sin\tfrac{\pi}{3}\right)$$

$$B\left(\frac{1}{2} \cdot \frac{\sqrt{2}}{2}, \frac{1}{2} \cdot \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right) \Rightarrow \boxed{B\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right)}$$

$$\cos \widehat{AB} = \cos \angle AOB = \cos(\vec{OA}, \vec{OB}) = \frac{\vec{OA} \circ \vec{OB}}{\|\vec{OA}\| \|\vec{OB}\|} = \vec{OA} \circ \vec{OB}$$

$$\Rightarrow \cos \widehat{AB} = \left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{4}, -\frac{1}{2}\right) \circ \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{12}}{16} - \frac{\sqrt{12}}{16} - \frac{\sqrt{3}}{4}$$

$$= \frac{-8\sqrt{3}}{16} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \widehat{AB} = \arccos \frac{-\sqrt{3}}{2} = \boxed{\frac{5\pi}{6}}$$